PROCEEDINGS OF THE 47th CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Auckland
Aotearoa New Zealand
July 17-21
2024

EDITORS
Tanya Evans
Ofer Marmur
Jodie Hunter
Generosa Leach
Jyoti Jhagroo

VOLUME 2
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RETHINKING MATHEMATICS EDUCATION TOGETHER

Cite as:

Evans, T., Marmur, O., Hunter, J., Leach, G., & Jhagroo, J. (Eds.) (2024). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2). Auckland, New Zealand: PME.

Website: https://events.massey.ac.nz/pme-47-conference/

Proceedings are also available on the IGPME website:
http://www.igpme.org

ISBN: 978-1-0670278-2
ISSN: 0771-100X

Logo designed by Jason Lamontanaro

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MIGRANT STUDENTS' PERCEPTIONS OF EXPERT SYSTEMS IN MATHEMATICS CLASSROOMS IN CANADA

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International migration flows have had a growing impact on mathematics classrooms in many parts of the world. Research suggests that many students from immigrant backgrounds face challenges in the learning of mathematics. We present findings from a study designed to explore how migrant students experience mathematics classrooms in Canada. We utilised Bauman's notion of expert systems to analyse how migrant students position themselves with respect to authorities in mathematics classrooms. Our findings show students do perceive elements of the expert systems, which hinders the feeling of equality and inclusion in the mathematics classroom, resulting in three tensions. These tensions allow us to recognise the structural constraints within which migrant students operate and the perceived opportunities for multiplicity of expertise.

Many studies in the field of mathematics education have underscored the difficulties that migrant students often face as they try to navigate the practices and interactions of mathematics classrooms in their new setting (*e.g.*, Civil & Planas, 2010; Takeuchi, 2019). These challenges are of increasing significance in the many countries experiencing an increase in immigration. In Canada, for example, 30 percent of schoolchildren are either immigrants themselves or have at least one parent born abroad (Statistics Canada, 2022). Such students all, to a greater or lesser extent, experience a change in mathematics teaching and learning. They may bring novel ways of doing and understanding mathematics (*e.g.*, algorithms, specific mathematics content) and may encounter ways of doing and being taught mathematics with which they are unfamiliar. If mathematics teachers are to support migrant students as they adapt to a new culture of mathematics teaching and learning, research is needed to understand these students' perspectives and experiences of mathematics classrooms in the new context. In this research report, we present research that examined how migrant students perceived aspects of the organisation of mathematical authority and expertise.

LITERATURE AND THEORETICAL FRAMEWORK: EXPERT SYSTEMS, MATHEMATICAL AUTHORITY AND INCLUSION

In his work to understand the social production of dependency, Bauman (1992) conceptualised what he called expert systems. Bauman assumes that life in society is unimaginable without a set of skills which enables each individual to interact with others and gain social affirmation. It follows that in such a society, there will be experts who know these skills best. Bauman assumes an expert system to have five

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1–8). PME.

characteristics: 1) doing things properly requires particular knowledge; 2) such knowledge is distributed unevenly; 3) those who have it ought to be in charge; 4) being in charge carries responsibility for how things are done; and 5) for others, personal responsibility rests entirely on following the advice of experts. Within an education system, we can various experts of various forms, such as, for example, the mathematics curriculum, which is a document for others to follow. The characteristics of the expert system, comprising the five identified elements, could also be found in mathematics teacher education programs, in the way they make assumptions about the role of a teacher in mathematics classrooms. Even mathematics itself can be considered as an expert system.

More specifically, we can assume the existence of a requisite set of knowledge and skills to function in a mathematics classroom, with certain versions of the five characteristics considered valid. These necessary skills include appropriate mathematical knowledge, alongside social and behavioural competencies. While the conceptualisation of responsibility may pertain primarily to the teacher's role, it also extends to encompass certain skills expected of students. Hence, it might seem obvious that expert systems are smoothly intertwined in the activities of mathematics classrooms, so much so that one could even ask: why not? Or what else? In this research report, our aim is not to re-state the obvious. Instead, our focus is on considering the accounts of migrant students regarding their experiences in mathematics classrooms, which parallel the five elements of an expert system. Through this exploration, we seek to illuminate how the perceived existence of expert systems by migrant students may impact notions of equity and inclusivity, particularly for migrant students entering a "new" mathematics classroom as they strive to find their place within it.

Theoretically, in an expert system, we see that when the expert (such as a mathematics curriculum) performs with languages and features of authority, and when others (such as teachers and learners) accept and follow the language of authority, then uniformity might become an attribute of the entirety of the system. That is, on the basis of the five characters of an expert system, all elements of the system - the expert, the area of expertise, and the learner - act with a sense of conformity (Neyland, 2010). In our study, we inquire about the potential impact of a perceived 'uniformity' and 'conformity' on the notions of inclusion and equity within the classrooms. Acknowledging the hypothetical nature of this question, we posed it based on the premise that migrant students' experiences might reflect the elements of an expert system and the possibility that authoritative influences within the system could impact the sense of inclusivity.

The uneven distribution of expertise in mathematics classrooms raises the question of where mathematical authority lies. Several researchers refer to the concept of mathematical authority (Amit & Fried, 2005; Langer-Osuna, 2017). Amit and Fried (2005) suggest that, in classroom interactions, often only a few are seen as having mathematical expertise and thus as having mathematical authority within a classroom.

Yet, a key component of collaborative mathematics classrooms is that mathematical authority in distributed: students do not necessarily see that mathematics knowledge and expertise reside solely in the teacher, textbook, or curriculum but see themselves as mathematically capable and as possessing mathematical authority (Cobb *et al*., 2008; Langer-Osuna, 2017, 2018). However, many factors come into play that may limit this distribution of mathematical authority such as the role of power within the social setting of a classroom (Langer-Osuna, 2017) which may influence who has a voice and consequently who is seen as having mathematical authority.

Promoting the inclusion of multiple voices in mathematics classroom activities takes us to another body of literature. Adopting a broadly resource-oriented perspective, mathematics education research includes countless studies that have invited the cultures and perspectives of migrant communities into innovative approaches to the teaching and learning of mathematics. Theoretical perspectives include culturally responsive teaching, funds of knowledge (*e.g*., González *et al*., 2009) and ethnomathematics (*e.g*., D'Ambrosio, 2006). If these theories have a common thread, it is the importance of more inclusive forms of teaching and learning mathematics, through utilising the cultural, historical and linguistic resources of migrant students. Such attention to the potential, capacity and resources of migrant students is important, because research studies often have underscored the difficulties that migrant students often face as they try to navigate the unfamiliar practices and interactions of mathematics classrooms in their new setting (*e.g*., Planas & Civil, 2010; Takeuchi, 2019). These theoretical frameworks and research projects share a foundational commitment to acknowledging and the novel ways and unique perspectives migrant students bring to the learning and understanding of mathematics. Inclusive and equitable mathematics classrooms. However, a dilemma arises. On the one hand, the mathematics education research strongly advocates for including a multiplicity of ways of doing and being in mathematics classrooms, embracing a variety of approaches and ways of engaging with mathematics. On the other hand, the potential existence of perceived elements of expert systems in the mathematics classroom introduces a concern about the emergence of a sense of uniformity and conformity. In this research report, we present findings concerning (i) experiences of students of the learning and teaching of mathematics in the context of migration; and (ii) migrant students' perceptions of the system of mathematics education they have to navigate and be accountable towards. Our research questions are therefore:

- If and how do migrant children experience characters of the expert system in mathematics classrooms?
- If they do perceive such characters, how do they position themselves/assume responsibilities, in relation to these systems?

RESEARCH DESIGN AND METHODS

The work we present in this research report is from a larger study which sought, among other things, to give voice to students by locating their knowledge, expertise, and experiences at the forefront of the research. The study utilises the voices of the children and the teachers, to highlight "unfamiliar" mathematics for teachers and for students.

We report on the first phase of data collection, in which migrant students in grades 4–12 were invited to write a hypothetical letter about their experiences in mathematics classrooms in Canada. The letter could be addressed to a relative who will soon arrive in Canada, or to the teacher who will welcome the relative. We used SurveyMonkey to obtain participants' consent, obtain geographic information, and for participants to share their letters. Participants were also invited to take part in a follow-up interview.

We collected 57 letters and conducted 10 interviews with students who had been living in Canada for 1-5 years. Contributing students came to Canada from countries spanning the globe including, the Caribbean, China, Czech Republic, Hong Kong, Iran, Jordan, Lebanon, Morocco, Nigeria, Saudi Arabia, Turkey, United Kingdom, Vietnam, and USA. Other students shared having moved from one region to another within Canada. The letters were collected in 2022 and 2023. Through our initial reading of the data, we noticed the kind of power the participants attributed to the system and hence to expertise and mathematical authority. The letters were analysed using the five characteristics of an expert system in order to identity what kinds of expertise were apparent in students' letters, as well as to examine how students positioned themselves with respect to these systems.

RESULTS

Our results show that participating students do perceive behaviours and interactions in their new mathematics classrooms that resemble the five elements of an expert system. For example, they perceived that to do mathematics properly, they need to have particular knowledge, or they need to know a certain language. Further, they perceived that there are different authorities who possess knowledge and therefore who ought to be in charge, such as the teachers and the curriculum. Finally, they underscored their own responsibility and accountability, as a learner, in the system. In the following, we expand on these three points with illustrative extracts from the letters.

Particular knowledge: The area of expertise

Perhaps not surprisingly, different elements of mathematics emerge as areas of expertise. Some examples are multiplication, addition, and graphs. Letter 9, for example, describes the "math concept" as an area of expertise that is universal but some concepts come in different grades in Canada compared to China. They state:

> As a student who transferred schools from different countries like you, I believe that math concepts are universal. It's a matter of when the content is taught. For the same content, some countries might introduce it at a lower grade while others teach it at a higher grade.

This student treated "math concepts" as an area of expertise that is taught in mathematics classrooms. The only catch is that "some countries might introduce it at a lower grade while others teach it at a higher grade". Other students mention other areas of expertise, including *language*, *word problems*, *mathematical terminology*, and *formulas*. Contrary to mathematical formulas and knowledge, they find language a big challenge. Letter 9 continues that their "biggest barrier was language. Being unable to understand English, [they] could not comprehend what the word problems asked for." Letter 15 explained: "math in Canada [...] is in English, which is very weird at first but you get used to it. Not many of the concepts are different as long as you are able to understand it properly".

Common among the extracts mentioned above and in almost all letters, students perceive mathematics as an area of expertise. At one level, this seems obvious, as they are participating in a mathematics classroom to learn mathematics. But what we noticed in our analysis is how the students attribute the areas of expertise as being defined by the experts, and how mastering these areas is the responsibility of the students. They not only perceived these areas of expertise, but also perceived that they come with clear expectations with regard to execution and the 'proper' way of doing things. These expectations are mentioned by students in different parts of their letters. We explain more in the next section.

Uneven distribution and authority: The experts

In their letters, students allocate the role of expert to actors such as the government, teacher, grades, lectures, tests, or the school system. Determining who is the expert at a given time is described as a dynamic relationship where the experts' roles are connected to each other and change based on the situation. While the teacher is the most knowledgeable person in the class as an expert, their role shifts to a person responsible to follow the advice of other experts, such as the curriculum. This dynamic relationship comes from the perception that there exists more than one expert system in the class or school working interconnectedly. That is, knowledge is not evenly distributed between experts as well, so there are different responsibilities (power) in different expert systems. Letter 19 mentions mathematics curriculum and the teacher as two elements of a mathematics classroom in two different expert systems. The author says:

> I would always ask the teacher to rephrase the question [...] I had never heard any of the mathematical terms before, yet all of a sudden I have to know them [...] Since the formulas and knowledge are the same across the world, it would not be too much difference other than the language [...] Particularly, my middle school math teacher provided square and rectangle tiles to help students learn about perimeter and area visually.

Here, two different roles of the teacher in the class are recognised from the student's perspective. One role is as an expert who is knowledgeable and responsible for their students and acts independently of the curriculum when the student "ask[s] the teacher to rephrase the question". The second role of the teacher is that of a person who is responsible for what the expert (in this example the content of the curriculum) tells her to do, such as providing "square and rectangle tiles to help students learn about perimeter and area visually".

Responsibility to follow the experts: The non experts

Students positioned themselves in the expert systems that they have perceived. Participating students generally assumed personal responsibility to follow the advice of the experts, such as the teacher or the demands of the curriculum. We also noted that the way that students position themselves within these expert systems informed their reported actions when participating in the mathematics classroom. For example, as part of their responsibility, students mentioned "putting in great effort", "having good time management", "trying hard to understand", "practicing carefully", "answering correctly", "working harder", and "doing homework". Letter 9 states:

> All in all, math is a fascinating subject. Do not be afraid of any difficulties, eventually, you will overcome all of them with the great effort you put in.

In letter 15, a student describes their responsibility in "get[ting] used to [English]", "understand[ing] it [mathematical concepts] properly", and "time management". They explain:

> Not many of the concepts are different as long as you are able to understand it properly. The only difference is that you need really good time management here. So as long as you can do that, you'll be fine. Good luck in school.

Doing homework is another responsibility that students assume as a practice asked by the experts. The author of letter 5, for example, compares mathematics classrooms in China with Canada and explains:

> I think China's math is more strict than Canada's. In China we did a lot of exercises in one chapter. By doing all the practises the teacher posted and finishing all the homework carefully students can get good grades.

These descriptions emphasise the distinctions between personal and imposed responsibility among students. Their descriptions underscore a personal connection to both the expert and the field of expertise, as well as their individual preferences, commitment, and efforts.

Illustratively, in letter 5, a student positioned themselves in relation to the expert system which imposes the expectation of obtaining "good grades". The student suggests that by diligently completing "all the practices the teacher posted" and finishing "all the homework carefully", they can achieve the desired outcome of "good grades". These instances serve to delineate when an action is perceived as a personal responsibility, stemming from individual choices, preferences, and efforts. In contrast, responsibility is also depicted as being guided by external requirements, such as the explicit demands of the expert system.

Our analysis of all the letters reveals a common understanding: when exploring the students' positions, actions are articulated as manifestations of personal responsibility directed towards both the experts and the areas of expertise. This differentiation elucidates how students navigate their roles based on intrinsic motivations and choices, as well as external expectations imposed by the expert system.

DISCUSSION

We have presented the outcome of our exploration of how migrant students experience mathematics classrooms, with a focus on the distribution of authority and the existence of expert systems. Our findings show that migrant students perceived interactions and behaviours that could be characterised as elements of different expert systems within the mathematics classroom. That is, they assumed expert knowledge that they need to acquire, such as mathematics and the language of instruction. They perceive actors such as teachers, curricula and at times the education system to be experts (*i.e.*, the entities that have the desired knowledge). They also perceived a certain position within these systems within which they assumed responsibilities. For example, they assumed that in order to gain the desired knowledge they need to listen to the teacher, ask questions, work hard, follow the instruction, learn the language of instruction and so on.

In the context of expert systems, actions are framed and delineated in terms of certain authorised procedures. Such framing of action introduces tensions in students' sense of responsibility and accountability towards the system. The first tension is about students' perception of their responsibilities in order to fulfill the demands outlined by the expert systems. The demands of mathematical authority within the system have the effect of reducing personal responsibility to rule-following and adherence to procedures, which goes against any attempt to promote a greater distribution of mathematical knowledge. The second tension arises from the expert system's inclination to validate specific knowledge, potentially dismissing multiplicities of knowledge and approaches. Our findings showed that almost all students perceived the mathematics taught in the classroom as the 'proper' knowledge which they are required to learn. This is particularly important because learning the requisite knowledge of the expert risks the monopolisation of validity as defined by the system. That is, what counts as proper knowledge is the knowledge defined by the system and not alternative knowledge not defined by the system, such as, potentially, migrant students' ways of knowing mathematics. The third tension involves the level of autonomy and independence that students perceive in their actions and interactions. The five characteristics of an expert system construct the learners of mathematics as not entirely self-sufficient, again fostering a tendency towards conformity.

We believe that the three tensions described above directly underscore issues of equity and inclusion in the mathematics classroom and affect inclusion. The migrant students' letters illustrate how the goal of inclusive and equitable participation of migrant students in mathematics classrooms takes place, for the students, in a context in which teachers and students must align themselves within expert systems, adhere to prescribed requirements, and demonstrate expected skills. Building on these tensions, further research is required to consider their effect on migrant students' navigation of and positioning in the mathematics classroom and to explore the ways in which broader spaces could be conceived for the voices of migrant students to be heard and included.

Acknowledgments

The analysis presented in this research report is from the *Migration in Mathematics Classroom* project, funded by the Social Science and Humanities Research Council of Canada, grant number 435-2018-1310.

References

Amit, M., & Fried, M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th grade classroom. *Educational Studies in Mathematics, 58*, 145–168.

Bauman, Z. (1992). Life-world and expertise: Social production of dependency. In N. Stehr, & R. V. Ericson (Eds.), *The culture and power of knowledge: Inquiries into contemporary societies* (pp. 81–106). Walter de Gruyter.

Cobb, P., Gresalfi, M., & Hodge, L. l. (2008). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 39*(1), 1–29.

D'Ambrosio, U. (2006). *Ethnomathematics: Link between traditions and modernity*. Sense.

González, N., Moll, L. C., & Amanti, C. (2009). *Funds of knowledge: Theorizing practices in households, communities, and classrooms*. Routledge.

Langer-Osuna, M. J. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education, 48*(3), 237–247.

Langer-Osuna, M, J. (2018). Exploring the central role of student authority relations in collaborative mathematics. *ZDM Mathematics Education, 50*, 1077–1087.

Neyland, J. (2010). *Rediscovering the spirit of education after scientific management.* Brill.

Planas, N., & Civil, M. (2010). Discourse processes in critical mathematics education. In H. Alrø, O. Ravn, & P. Valero (Eds.), *Critical mathematics education: Past, present and future* (pp. 145–159). Sense.

Statistics Canada (2022). School attendance by visible minority and immigrant status: Canada, provinces and territories, census metropolitan areas and census agglomerations with parts. https://www150.statcan.gc.ca/t1/tbl1/en/tv.action?pid=9810043301

Takeuchi, M. A. (2018). Power and identity in immigrant parents' involvement in early years mathematics learning. *Educational Studies in Mathematics, 97*(1), 39–53

PROPOSAL FOR THE STUDY OF MATHEMATICS TEACHERS' BELIEFS BASED ON THE ANALYSIS OF THEIR ACTIONS

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In recent years, research on mathematics teachers' beliefs has increasingly focused on their relationship with the instructional practices of teachers. This article proposes a model that enables the study of mathematics teachers' beliefs through comprehensive analysis of their actions in the classroom. The proposal incorporates the notions of teachers' actions, norms and metanorms of the Ontosemiotic Approach as its articulating axis. Furthermore, this paper presents an example of a practical implementation of these levels of analysis in the study of a prospective teacher's beliefs, showing the viability of this model.

RATIONALE

Some of the aspects currently demanded by the research in the relationship between beliefs and practices of mathematics teachers are investigating the roots of the observed practices; knowing which beliefs affect specific practices; as well as ensuring that the research results serve as basis for developing more efficient training programs (Goldin et al., 2016). Although there are theoretical and methodological frameworks connecting teachers' beliefs and teachers' actions (Leatham, 2006; Schoenfeld, 2000), the existing literature fails to present frameworks that facilitate the execution of studies that combine the following characteristics: begin with an analysis of teachers' actions; enable the examination of enacted and professed beliefs; yield results that offer insight into the implications of these beliefs in the actions that teachers carry out in the classroom; and enable the identification of lines of action to improve such actions.

This paper presents a proposal of levels of analysis for a class episode that enables the study of mathematics teachers' beliefs. The goal is to contribute and advance in the achievement of the aforementioned points. To do this, we propose to analyse the class episodes through the teacher trajectory, and the norms and meta-norms put into play; conduct interviews with teachers to reflect on class episodes; and consider as units of analysis, the actions that are in accordance, or not, with didactic suitability criteria.

THEORETICAL FRAMEWORK

For this proposal we have considered the relationship between beliefs and teaching actions proposed by (Schoenfeld, 2008) and (Cobb & Yackel, 1996). In the first case, it is said that beliefs give rise to goals that teachers establish. These goals are studied by researchers who use this theoretical framework by identifying the actions carried out by teachers, seeking to find in them what the teachers propose to the students, explicitly or implicitly. Those proposals are directly related to what is known in the

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 9–16). PME.

OntoSemiotic Approach (OSA) as "teaching actions". In such a way that the study of teaching actions, resultos an ideal element to investigate the beliefs that give rise to them. Furthermore, Cobb and Yackel (1996) identify the relationship between beliefs and sociomathematical norms, mentioning that beliefs refer to an individual's understanding of normative expectations, while social norms are thought of as shared beliefs. In this case, OSA's normative dimension is useful to study teachers' beliefs.

In this way, we consider that OSA is a theoretical framework that can shed light on the study of beliefs through the detailed analysis of teaching actions. Below we describe each of these notions.

The Ontosemiotic approach (OSA)

OSA proposes the analysis of the didactic configurations, which means the study of the evolution of the epistemic trajectory proposed by the teacher and the cognitive trajectory developed by the students; as well as the interactions between teacher and student, given by the web of responsibilities in charge of each of them (teacher and student trajectory); in addition to the distribution of the resources used (mediational trajectory) and the emotional states in relation to the instructional process (affective trajectory).

Nevertheless, it is essential to recognize that when referring to didactic configurations, we are referring to the classroom in its entirety. This implies that the actions of both teachers and students, as well as their relationship, are analysed. However, some of the trajectories enable us to concentrate more specifically on particular aspects. In particular, the "teaching trajectory" enables identification of teaching actions by classifying them based on their function (Godino et al., 2006):

Motivation. Those related to the creation of a climate of affection, respect, and encouragement for individual and cooperative work, so that it is involved in the instructional process.

Assignment of tasks. It refers to the practices carried out by the teacher to direct and control the process of study, assign times, adapt tasks, orient, and stimulate the students' functions.

Regulation. It involves practices related to the establishment of rules, the recall and interpretation of previous knowledge necessary for the progression of learning and the readaptation of the prepared planning.

Evaluation. Those practices in which the teacher observes and assesses the state of the learning achieved at critical moments (initial, final, and during the process) and resolution of the individual difficulties observed.

Since analysing an instructional process requires understanding "the rules of the language game in which it takes place" (D'Amore et al., 2007), OSA, through its normative dimension, analyses the systems of rules, habits and norms that restrict and support didactic and mathematical practices in the processes of studying mathematics.

In instructional processes certain norms are negotiated between teachers and students. These normative expectations are communicated through their actions, implicitly or explicitly. Thus, when teachers offer students a specific problem-solving procedure, promote a particular way of working, or encourage the use of a particular type of material, they are laying the foundations for a particular mode of engagement within that instructional process.

Since these rules are negotiated within the instructional process, some of them will be discarded quickly, others will have a longer negotiation process, and some will remain established and will be shared by the participants for a certain time. In this way, the "rules" in OSA reflect performance expectations and obligations that each participant considers for himself or for the rest (Planas & Iranzo, 2009). However, if the rules remain unchanged for a period of time, they are considered metanorms. According to the facets of the instructional process, the norms are classified as follows (Godino et al., 2009):

Epistemic Norms. The epistemic norms are those that regulate the teacher's work in relation to mathematical knowledge. They contemplate the systems of practices put into play in the class, as well as their decomposition into problem-situations, languages, properties/propositions, procedures, concepts/definitions and arguments.

Cognitive norms. They regulate the work of students in relation to mathematical knowledge. For the study of the beliefs of teachers of mathematics, beyond being interested in what the students' practices are, it is interesting to know which of these practices the teacher validates and promotes.

Interactive norms. These rules refer to the interactions between the teacher and the students, as well as between the students themselves. These rules allow us to identify what responsibilities each participant is assuming regarding the mathematical practices that are carried out in the instructional process.

Affective norms. Those that regulate the affectivity of the people involved in the instructional process.

Mediational norms. They regulate the use of technological and temporal resources, these norms are related to the mediational trajectory, since it is interesting to know the rules that the teacher tries to establish regarding the allocation of times and the use of materials and resources for the teaching and learning of mathematics.

Ecological norms. They regulate the relationship with the environment in which the instructional process takes place (considering social, political, economic, curriculum factors, etc.).

As mentioned previously, some of the rules will have more relevance in the instructional processes since they imply a certain regularity. In OSA the analysis of regularities in norms leads to the notion of metanorm, which can be understood as norms that apply to other norms or norms that remain unchanged during a certain period of time, and that become a part of every set of norms during that period, even if they are not consistently followed or properly executed afterwards. In this regard, the OSA proposes a typology for these metanorms, which is outlined below (D'Amore et al., 2007):

Epistemic metanorms. They regulate epistemic norms. They respond to how the mathematical objects are put into play in the instructional process.

Instructional metanorms. Are the metanorms that regulate every norm related to the teaching of mathematics.

Cognitive metanorms. Are the metanorms brought by the students concerning the mathematics to be learned (mathematical metacognition) and how they are learned (didactical metacognition).

According to Schoenfeld (2008) and Cobb and Yackel (1996), the actions carried out by teachers, the norms and meta-norms derived from these actions, constitute elements that allow us to study beliefs of mathematics teachers.

However, we agree with Speer (2005) regarding the need to study both enacted and professed beliefs, so, in addition to this analysis, we propose interviews with teachers to analyse video recordings of their own classes, which allow us to have a scenario and common language between researcher and teacher.

In order to improve the instructional processes, researchers might be interested in trying to understand in depth those practices that are in accordance with some curricular proposal or with a certain teaching model. This is where the beliefs of teachers become relevant, since they allow us to understand the reasons that led teachers to perform such actions.

OSA, through the Didactical Suitability tool (Godino et al., 2023) evaluates the instructional processes, allowing to distinguish possible improvements in them. This tool provides "general principles and criteria based on research-proven results for which there is consensus in the corresponding scientific community" (Godino et al., 2023, p. 114), so we consider that the combined study of the instructional processes, based on this tool and the beliefs of mathematics teachers, are a fruitful scenario to establish lines of actions that may serve as a basis for efficient teacher development programs.

MODEL FOR THE STUDY OF TEACHERS' BELIEFS THROUGH THE ANALYSIS OF THEIR ACTIONS

For our proposed study of teachers' beliefs based on their practices, we propose a video recordings analysis model of an instructional process, which is based on the application of five stages that are described below:

Level 1. Study of teaching trajectory

Level 2. Identification of norms

Level 3. Identification of metanorms

Level 4. Inference of beliefs / Generation of interview script

Level 5. The study of beliefs through reflection on their actions

The first three levels are descriptive, they seek to answer what mathematics is promoted in the classroom and how its teaching is managed. The first level of analysis involves the identification of mathematical practices promoted by the teacher, through the epistemic configuration promoted; as well as the identification of the instructional interventions that allowed to manage the instructional process, which is possible to make through the analysis of the teaching trajectory.

The second level requires identifying certain regularities in the previous level, as well as their relationship with the other facets. Although making conclusions about the normative nature of a practice implies the observation of several class episodes, we consider that the model can be applied for short episodes, so it can be limited to observing the normative intentionality of certain practices (Planas & Iranzo, 2009).

Starting from the previous level, the third level seeks to identify prevailing norms over time or specific characteristics within them, corresponding to the metanorms associated with those norms. For example, if certain arguments are normed, the metanorms would assist in identifying their characteristics that respond to how those arguments are presented.

The intention of the fourth level is to generate instruments for the reflection that will take place at the next level. Once there are episodes within the instructional process in which norms and metanorms that are of interest are distinguished, the researchers generate video fragments in which these norms and metanorms are evidenced. The interview script that is sought to be generated at this level is intended to analyse with the teachers the norms and metanorms that they have tried to establish in the classroom. In such a way that the questions generated by the researchers revolve around the reasons that the teachers have, to promote these norms and metanorms, which will constitute the beliefs of the teachers. At this level it is possible to infer certain beliefs, which are known in the literature as enacted beliefs. However, these enacted beliefs need to be contrasted with the teacher at the next level of analysis.

Finally, the fifth level serves a dual purpose of contrasting and of expanding. Its goal is to reflect with teachers on the norms and metanorms identified in the first levels, using video fragments of their instructional process to observe these notions. Through this reflective process with the teachers, the researcher aims to contrast the beliefs he/she inferred at the previous levels. Additionally, this may bring forth beliefs expressed by the teachers which the researcher may have not previously considered.

Units of analysis for the study of beliefs of mathematics teachers

En every class we may find class episodes that favour or hinder the instructional process as long as they are aligned or goes against the suitability criteria (Godino et al., 2023), respectively. In this way, we consider that each of these episodes could be considered as a unit of analysis of the instructional process useful for the study of the beliefs of mathematics teachers.

EXAMPLE OF THE APPLICATION OF THE MODEL

We describe the analysis of an episode in which the levels of analysis described in the previous section were implemented. It is an algebra class, which was part of a microteaching process. The participants are preservice teachers who are in the last semesters of the career for high school Mathematics teachers. The only instruction by researchers was to develop the learning goal "Show that they understand the quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ recognizing the quadratic function in daily life situations and other subjects; representing it in tables and graphs manually and/or with educational software; determining special points on its graph; selecting it as a model of quadratic change situations in other subjects, in particular supply and demand" (MINEDUC, 2023, p. 31).

The unit of analysis was determined considering the component "Propose definitions and procedures that are clear, correct and adapted to the educational level to which they are addressed" (Godino et al., 2023, p. 18), from the criteria of suitability for the epistemic facet. Next, the levels of analysis are applied to the a class episode in which it is observed that the teacher makes an error when mentioning that $f(x) = (ax + b)(cx + d)$ is quadratic, only if it is written into the form $f(x\ ) = ax^2 + bx + c$.

Level 1. Study of teaching trajectory

At first, the teacher asked if $f(x) = (2x + 3)(x + 1)$ was a quadratic function and validated the students' response “if we solve it (referring to developing the product of binomials), then it will be a quadratic function.” The same thing happened for $f(x) = (x + 2)(3x - 1)$.

In a second moment, the teacher asks the students to evaluate the function $f(x) = (7x - 2)(9x - 3)$ for $x = 1$ and when the student tries to perform $f(x) = (7(1) - 2)\ (9(1) - 3)$ the teacher forbids it, mentioning that he must first transform that expression into a quadratic function.

Level 2. Identification of norms

The epistemic norms found in these episodes are: $f(x) = (2x + 3)(x + 1)$ is not a quadratic function, but $f(x) = 2x^2 + 5x + 3$ is. To transform a factored function into a quadratic it is necessary to develop the product of binomials.

Level 3. Identification of metanorms

From these norms, we may conclude that a quadratic function must be in the standard form and if it is expressed in the factored form, it is not a quadratic function.

Level 4. Inference of beliefs / Generation of interview script

Given this, it was inferred that the teacher believed that the factored form and the standard form of the quadratic function were different and that only the standard form was valid. We asked her why she had not allowed the student to evaluate $x = 1$ in the factored form.

Level 5. The study of beliefs through reflection on their actions

The teacher mentioned that she knows that the factored form represents a quadratic function and that, if the student had evaluated in it, it would have given the same answer as evaluating in the standard form. However, she does not want to allow students to do this, since "right now they are learning the form $f(x) = ax^2 + bx + c$, so they must write it and work with the function that way" and adds "Writing the function as $f(x) = ax^2 + bx + c$ is more useful, because it allows you to identify the coefficients and, with it, the behavior of the function."

DISCUSION

Considering this class episode, based on the suitability criteria, allowed us to identify the beliefs that lead the teacher to establish characteristics of the quadratic functions that are not correct, by establishing that the factored form is not a quadratic function. The inferred belief that the teacher considered that only the standard form represented the quadratic function was incorrect; however, it allowed us to identify that the belief in the usefulness of the standard form of the quadratic equation takes precedence over the belief that both forms represent a quadratic function. For her, it is important that this is the form that is being learned; furthermore, she believes that this form has advantages that other notations do not have. Possibly working with another notation might not be convenient or beneficial for students' learning, which leads her to restricting its use.

In this case, we propose that, a teacher development program should modify these beliefs, resulting in the teacher showing the usefulness of each representation instead of establishing it in the way in which she does it, since it can lead to obstacles in student learning. Or, at least, not say that this notation does not represent a quadratic function.

CONCLUSIONS

In this paper we have presented an analytical model that can serve as an alternative for the detailed study of the beliefs of mathematics teachers through the analyses of their actions. Due to space limitations, we have limited ourselves to showing normative intentions that were identified in a class episode. While the proposed model also enabled the identification of teachers' beliefs, a larger number of class observations would provide greater clarity on the identified norms and metanorms. Thus, although it is not a limitation, it is appropriate and advisable to have extended periods of class observations.

This proposal of levels of analyses enables not only the identification of teachers' beliefs but also the implications of those beliefs on the actions that teachers carry out in their classrooms, and some basis for efficient teacher development programs.

Acknowledgements

This research is part of the FONDECYT regular research project 1200005 supported by the National Research and Development Agency (ANID) of Chile.

References

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190. https://doi.org/https://doi.org/10.1080/00461520.1996.9653265

D'Amore, B., Font, V., & Godino, J. (2007). La dimensión metadidáctica en los procesos de enseñanza y aprendizaje de la matemática. *PARADIGMA*, *28*(2), 49–77.

Godino, J. D., Batanero, C., & Burgos, M. (2023). Theory of didactical suitability: An enlarged view of the quality of mathematics instruction. *Eurasia Journal of Mathematics, Science and Technology Education*, *19*(6). https://doi.org/10.29333/ejmste/13187

Godino, J. D., Contreras, Á., & Font, V. (2006). Análisis de procesos de instrucción basado en el enfoque ontológico-semiótico de la cognición matemática. *Recherches En Didactiques Des Mathematiques*, *26*(1), 39–88. https://revue-rdm.com/2006/analisis-de-procesos-de/

Goldin, G. A., Hannula, M. S., Heyd-Metzuyanim, E., Jansen, A., Kaasila, R., Lutovac, S., Di Martino, P., Morselli, F., Middleton, J. A., Pantziara, M., & Zhang, Q. (2016). Attitudes, Beliefs, Motivation and Identity in Mathematics Education. In G. Kaiser (Ed.), *ICME 13*. Springer International Publishing.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education*, *9*, 91–102. https://doi.org/https://doi.org/10.1007/s10857-006-9006-8

MINEDUC. (2023). *Actualización de la priorización curricular para la reactivación integral de aprendizajes. Matemática*.

Planas, N., & Iranzo, N. (2009). Consideraciones metodológicas para la interpretación de procesos de interacción en el aula de Matemáticas. *Revista Latinoamericana de Investigación en Matemática Educativa, RELIME*, *12*(2), 179–213. https://www.redalyc.org/articulo.oa?id=33511498003

Schoenfeld, A. (2000). Models of the Teaching Process. *Journal of Mathematical Behavior*, *18*(3), 243–261.

Schoenfeld, A. (2008). On Modeling Teachers' In-the-Moment Decision Making. *Journal for Research in Mathematics Education*, *14*, 45–96.

Speer. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics*, *58*, 361–391.

IS IT A FRACTION, OR SHALL I DIVIDE IT?

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The fraction representation can symbolize different mathematical concepts because the symbol a/b is polysemic. Since students' difficulties in acquiring the different conceptual meanings that the fraction representation denotes are well documented in research, we examined two commonly used textbook series in Sweden concerning how the polysemic aspect is displayed by analyzing how fractions and division are introduced and how the first image of the concepts is challenged with other images. We found one-sided representations of fractions as part of the whole, division as quotients greater than one, and weak support for understanding the polysemic aspect of a/b.

INTRODUCTION

Consider the following question: Three children are to share a cake. How much cake does each child get? To solve the problem, you need to set up the division one divided by three, 1/3, which gives the solution one-third, 1/3. A rather strange situation now arises. On the left-hand side of the equal sign, 1/3 means the operation division. On the right-hand side, we have the answer to the question in fraction form. For clarity, we want to point out that division is the inverse *operation* of multiplication and that a fraction is a *representation* of a number—two different entities of mathematics.

$$\frac{1}{3} = \frac{1}{3}$$

Figure 1: The calculation and the solution

The two symbolic expressions are precisely the same but have different meanings because the fraction notation a/b, where a and b can be any numbers or expressions, except b=0, is a polysemic symbol. Polysemy is present whenever mathematical patterns identified in different circumstances share the same structure. This structure will then typically be subsumed under the same mathematical symbolism, in this case, a/b. The representation a/b subsumes, besides the operation division, part-whole, ratio, operator, quotient, and measure constructs of fractions, as well as all other expressions in quotient constructions. The finesse with the polysemic symbol a/b is that all quotient constructions, see Figure 2, follow the same mathematical rules despite having different meanings.

The hurdles of understanding fractions have been thoroughly researched. For example, Thompson and Saldanha analyzed how students perceive improper fractions, with 7/3 as an illustrative example. Under the assumption that students interpret 3/7 as part of a

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17–24). PME.

whole, 7/3 will appear highly suspect (Thompson & Saldhana) until students become versed in the polysemic nature of fractions (Ahl & Helenius, 2021b).

$$\frac{7}{4} \qquad \frac{1}{3} \qquad \frac{a/b}{a} \qquad \frac{\sqrt{3}}{2} \qquad \frac{\sum_{n=0}^{\infty} a_n X^n}{\sum_{n=0}^{\infty} b_n X^n} \qquad \frac{0.5}{6.7} \qquad \frac{\cos(x)}{\sin(x)} \qquad \frac{x^2+1}{2x^3-4x}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{a} = 1 \qquad \frac{a}{b} \cdot \frac{b}{a} = 1 \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$$

Figure 2: Examples of fraction representations and rules that apply

To counteract a one-sided view of fractions as part-whole construction, students need to work with different sub-constructs of fractions to differentiate between, for example, part-whole, measure, operator, ratio, quotient, and scale value. These different interpretations of fractions cannot be introduced simultaneously without causing confusion. Still, it is reasonable that improper fractions challenge the part-whole image to avoid a one-sided understanding of fractions as something less than one.

The understanding of improper fractions is hindered when part-whole fractions are introduced by the iconic representation of the shaded area of a circle, which only allows the experience of fractions as a number less than one. Because how do you represent 7/3 with a circle? It cannot be done. Students exposed to a long series of circle representations may have difficulties accepting that fractions could be greater than one. Since the understanding of 7/3 does not fall within the student's understanding of fractions as a part-whole construction, the alternative is to interpret 7/3 as two different numbers to be divided, especially if all the divisions so far in the students' life have been constructions where the numerator is greater than the denominator

The part-whole understanding of fractions may not appear problematic in the first years of schooling when the goal is often specifically to understand and operate on part-whole constructions. However, in line with Thompson and Saldhana (2003), a one-sided idea of fractions as part-whole representations may create obstacles to extending the concept to the polysemic universe that the fraction representation constitutes.

> To reiterate our point, the way students understand an idea can have strong implications for how, or whether, they understand other ideas. This observation is important for thinking about what students have learned or actually understand and it has implications for how instructional and curricular designers think about what they intend that students understand. (Thompson and Saldanha, 2003, p. 2)

Building on ideas to support progress in concept knowledge is the core of mathematics education. For progression in concept knowledge, students must learn that symbols can have different meanings in different situations, even when the notation is the same. Interpreting the meaning of polysemic concepts is necessary for students to undergo the epistemological shift required for them to move from creating meaning from situations (like equal sharing situations for division) and iconic schematic imagery (like part-whole figures for fractions) to creating meaning from symbol system relations

(Ahl & Helenius, 2021a; 2021b; 2022). If we want students to progress in concept knowledge, progression needs to be manifested in syllabuses, curriculums, and curricular resources such as mathematics textbooks. In this paper, we report on an investigation of how students are introduced to the polysemic aspect of fractions in two commonly used Swedish textbook series focusing on two cases:

RQ1: When are students introduced to improper fractions and division with numerators less than the denominator in two commonly used Swedish textbook series?

RQ2: How is the polysemic aspect of a/b conveyed?

THEORETICAL UNDERPINNINGS

Our theoretical framework will be articulated using the language of conceptual fields. Vergnaud (2009) outlines a concept as a triplet comprising three interconnected sets: the set of situations wherein the concept holds relevance, the set of operational invariants that an individual can utilize to address these situations, and the set of representations (symbolic, verbal, graphical, gestural, etc.) that can be employed to depict invariants, situations, and procedures. It is important to note that, in this definition, situations and invariants are considered psychological categories, representing mental constructs, while representations can encompass both mental and physical/external manifestations. Given our focus on analyzing progress in general mathematical concept knowledge rather than explaining the cognition of specific individuals, we will approach situations and invariants from an observer's standpoint, motivated by the fact that, within educational and mathematical contexts, a sufficient number of individuals will construct situations and invariants that are similar enough to justify discussing them as phenomena in their own right.

In previous research, we have described three theoretical pillars for progression in conceptual knowledge: 1) the origin of concepts, 2) the umbrella effect, and 3) the contradiction of invariants. (Ahl & Helenius, 2021b; 2022). Here, we use 1) and 2).

1). The origin of concepts: Vergnaud's (2009) conceptual framework posits an interconnection between situations, invariants, and representations. However, concerning concepts introduced in educational settings, the initial invariant serving as the foundation will typically originate from a situation or a representation rather than through a definition that explicates the concept through its formal relations to other concepts. Integrating this insight with our delineation of two types of representations, iconic and non-iconic, yields three fundamental approaches to conceptual generation. First, concepts can be associated with the invariants within a set of situations, such as when the concept of division is elucidated by specifying a quantity of items to be equally divided into a given number of bags. Second, concepts can be linked to iconic representations, exemplified by giving meaning to fractions through an image of a partially colored circle. Third, concepts can be tied to mathematical relations expressed in non-iconic symbol systems, as seen when division is defined by stating that a/b is a number c, such that $a = b \cdot c$.

2). The umbrella effect: Mathematical concepts are regularly subsumed into more general concepts. When concepts derived from situations or iconic representations are subsumed into more general concepts, certain invariants from the original concept may no longer remain invariant under the new overarching concept. For instance, in the context of equal sharing used to elucidate division, dividing *a* by *b* yields a number c that is smaller or equal to *a*. However, this does not hold for division in a general sense. Similarly, when part-whole relationships are represented iconically by circle sectors, no fraction can exceed the entirety of the circle. Yet, in the broader scope, *a/b* can assume any size—notably, all three examples in the preceding paragraph share the same symbol system, *a*/*b*. Although we may continue to distinguish between division and fractions in specific instances, from a mathematical standpoint, we can encompass both concepts under the broader umbrella of quotient constructions.

METHOD

From the theory described above, we singled out two methodological principles for the analysis: 1) To identify the origin of the concept of part-whole fractions, improper fractions, and division throughout two mathematics textbook series as our unit of analysis and to classify representations in terms of situational, iconic and symbol system representations; 2) To identify explanations for the umbrella effect, that is that *a*/*b* subsumes the polysemic concepts part-whole fractions, improper fractions, and quotients to divide.

Investigating two commonly used textbook series for grades 1-9 in Sweden, *Favorit Matematik* (TB 1) and *Matte Direkt* (TB 2), we first identified introductory instances where the textbook, as an agent for the authors, makes mathematical claims, argues for propositions or gives meaning to the concepts of part-whole fraction, improper fraction, and the operation division. For each such instance, we evaluated the instances in relation to points 1 and 2 described above.

RESULTS

In Table 1, we summarize the results of when the sub-constructs of fractions part-whole and improper fractions, as well as division with quotients larger than one and lesser than one are introduced in TB 1 and TB 2. We also identify instances of support and hindrance for perceiving the polysemic aspect of *a*/*b*.

Introduction of:	TB 1	TB 2
Part-whole fraction	Grade two (ages 7-8) Iconic representation of partly colored circles of a half, a third, and a fourth. No symbols	Grade two (ages 7-8) An iconic representation of colored circles that illustrate part-whole together with the symbolic representations

		of 1/2, 1/3, and 1/4unit fractions
Improper fraction	Grade five (ages 10-11) 7/2 = 3 1/2, illustrated with three colored circles and one half-colored circle	Grade seven (ages 12-13) 5/4 = 1 1/4 illustrated with one colored circle and a quarter of a circle
Division of whole number quotients larger than 1	Grade two (ages 7-8) Partitive division is introduced through situations of sharing parcels, and quotative division is introduced by situations of creating equal groups of people standing in line. The terms numerator, denominator, and quotient are introduced together with mathematical symbols. The fraction bar is not mentioned	Grade three (ages 8-9) Partitive division is introduced through the situation of distributing an equal number of fifteen spiders to three ghosts, that is, equal sharing. The iconic images of the spiders and ghosts are presented together with the symbolic representation of 15/3
Division of quotients lesser than 1	Grade five (ages 10-11) Division with 10, 100, and 1000 with symbolic representations	Grade seven (ages 12-13) Rewriting fractions in decimal representations, symbolic representations
Support for conveying the polysemic aspects of *a*/*b*	When improper fractions are displayed in grade five (ages 10-11), fractions are linked to division with a reminder as the method to change representation to a mixed fraction	It is explicit that fractions can be divided when a change in representation from a part-whole fraction to a decimal representation is introduced in Grade 7 (ages 12-13)
Hindrance for conveying the polysemic aspects of *a*/*b*	In grade three (ages 8-9), the *fraction bar* is introduced as the sign by which you recognize a fraction without mentioning division. In connection with division, the fraction bar is not mentioned until grade	The fraction bar is not named in any of the books in the series

	seven (ages 12-13), when it is called the *division sign*. In grade seven (ages 12-13), the fraction representation is incorrectly defined as rational numbers written in the form *a*/*b* where both the numerator and denominator are integers, omitting that *b* cannot be 0	

Table 1: The origin of the concepts of fractions and division and the umbrella effect of the polysemic expression *a*/*b*

DISCUSSION

Mastering the polysemic aspect of the representation of quotient constructions, *a*/*b,* gives opportunities to see connections between different mathematical areas. There is reason to believe that limiting representations and situations for introducing fractions and division hide the polysemic aspect of fractions (Ahl & Helenius, 2021b; 2022). Without knowledge of mathematical polysemy, some students may never realize the fact that 1/3 can both represent a number and a call to perform the operation division of 1 by 3. To investigate when students' images of fractions as part-whole relationships and division as an operation with a nominator larger than the denominator are challenged and how the polysemic aspect of *a*/*b* is conveyed, we examined two commonly used textbook series in Sweden.

In both textbook series, we found that part-whole fractions are introduced with iconic circle representations in grade two. The part-whole conception of fractions dominates until grade five in TB 1 and grade seven in TB 2 when improper fractions are introduced. Division is introduced through situations representing sharing and grouping. In grade two in TB 1 and grade three in TB 2. Although there are some differences between TB 1 and TB 2, we see the same pattern. The textbooks give a one-sided view of fractions as part of the whole, mainly using iconic circle representations. The part-whole understanding is never really challenged, as even when mixed fractions are introduced, it is done by extending the part-whole representation with additional wholes. Division is preferably expressed as quotients where the numerator is greater than the denominator. That division concerns quotients larger than one is only challenged when quotients are rewritten to decimal representation in Grade Five (TB 1) and Grade Seven (TB 2).

In relation to the symbol *a*/*b,* which we prefer to call the fraction symbol system, in none of the book series is there a serious effort made to describe the generality of the system and that the same manipulation rules apply regardless of if the symbol is interpreted as a division, a fraction or something else. The generality of the system involves that neither *a* nor *b* in *a/b* need to be whole numbers. They can be fractions, decimals, or many other mathematical entities. It is only for defining rational numbers that it needs to be ensured that a fraction can be written in the form *a/b* with *a* and *b* being whole numbers. As remarked by Thompson and Saldahna (2003), confounding fractions and rational numbers is a common tendency in textbooks. This confounding was present in TB 1. The first time general symbols denote fractions, the incorrect definition of fractions as rational numbers is presented. This carelessness can have fatal consequences for the students' image of fraction representation. They are deprived of the finesse of using manipulation rules for quotient constructions on any mathematical expression. We believe mixing up the definition of rational numbers with the description of the fraction symbol system representation is counterproductive for students' understanding of the polysemic aspect of the fraction symbol system. We also believe that the choice in TB 2 to exclude formal definitions of fractions entirely and not to name the fraction bar hinders students' understanding of the polysemic properties of *a*/*b*. Because how could something that doesn't even have a name contain important mathematical ideas?

We would like to remind the reader that we have only analyzed introductions of concepts in instances where the textbook, as an agent for the authors, makes mathematical claims, argues for propositions, or gives meaning to the concepts of part-whole fractions, improper fractions, and operation division. A mathematics textbook is full of problems with the potential to create meaning. We do not comment on the students' meaning-making in the work with the textbook. Ultimately, the teacher's organization of the teaching determines the students' meaning-making. However, since teachers may also have weak knowledge of the properties of fractions (e.g., Dreher & Kuntze, 2015), the need for adequate mathematical theory in textbooks increases.

The lack of explicit explanations in the textbooks may be because the authors rely on an abductive approach. Students are expected to expand their fraction concept with improper fractions just by being presented with it. The same can apply to understanding whether the student should regard a symbol *a*/*b* as a number or two numbers to be divided. Our point is that what is not said explicitly concerning the polysemic expression *a*/*b* may not be taught. While some students can create mathematical connections with little guidance, we firmly believe that clarity and explicitness about how mathematical concepts are subsumed under polysemic signs would benefit all students' development. Especially if you believe that progress in concept knowledge is to finally reach the epistemological shift where the meaning of concepts goes from residing in situations and iconic representations to residing in relationships in symbol systems (Ahl & Helenius, 2021a). Given the well-documented problems with understanding fractions that students have (Thompson & Saldhana, 2003; Niemi, 1996), we believe that mathematics education would benefit from explicit presentation of the fraction system, where the polysemic properties are at the center if students should be given opportunities to master to decide whether 7/3 is a fraction or if they should divide it.

References

Ahl, L. M., & Helenius, O. (2021a). A framework for analyzing progress in concept knowledge in mathematics textbooks. In G. A. Nortvedt, N. A. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future. Preceedings of Norma 20. The ninth Nordic Conference on Mathematics Education Oslo, 2021* (pp. 1–8). SMDF.

Ahl, L. M., & Helenius, O. (2021b). Polysemy and the role of representations for progress in concept knowledge. In Y. Liljekvist & J. Häggström (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF 12.* (pp. 101–110). SMDF.

Ahl, L. M., & Helenius, O. (2022). New Demands on the Symbols and Formalism Competency in the Digital Era. In U. T. Jankvist & E. Geraniou (Eds.), *Mathematical Competencies in the Digital Era* (pp. 159–176). Springer International Publishing.

Dreher, A., & Kuntze, S. (2015). PCK about using multiple representations – An analysis of tasks teachers use to assess students' conceptual understanding of fractions. In K. Beswick, T. Muir, & J. Fileding-Wells, J (Eds.), *Proc. 39th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 243–250). PME.

Niemi, D. (1996). Assessing Conceptual Understanding in Mathematics: Representations, Problem Solutions, Justifications, and Explanations. *The Journal of Educational Research, 89*(6), 351–363.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. W. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95–113). NCTM.

Vergnaud, G. (2009). The theory of conceptual fields. *Human development, 52*(2), 83–94. Harvard University Press.

Textbooks

Asikainen, K., Haapaniemi, S., Mörsky, S., Tikkanenen, A., Vehmas, P., Voima, J. (2018). *Mera Favorit matematik 2A*. Studentlitteratur.

Asikainen, K., Nyrhinen, K., Rokka, P., & Vehmas, P. (2018). *Mera Favorit matematik 3A, 3B, 4A, 4B, 5A, 5B, 6A, 6B*. Studentlitteratur.

Heinonen, M., Luoma, M., Mannila, L., Rautakorpi-Salmio, K., Tapiainen, T., Tikka, T., & Urpiola, T. (2018). *Favorit matematik 7, 8, 9.* Studentlitteratur.

Falck, P., Picetti, M. & Elofsdotter Meijer, S. (2011). *Matte Direkt Safari. 2A, 3A*. Sanoma Utbildning.

Falck, P., & Picetti, M. (2012). *Matte Direkt Borgen 4A, 4B, 5A,5B*. Sanoma Utbildning.

Carlsson, S., Falck, P., Liljegren, G., & Picetti, M. (2012). *Matte Direkt Borgen 6A, 6B.* Sanoma Utbildning.

Carlsson, S., & Hake, K. B. (2016). *Matte Direkt 7, 8, 9.* Sanoma Utbildn

EXAMINING STUDENT WELLBEING AND PARENTAL EDUCATIONAL ATTAINMENT IN A U.S. COLLEGE MATHEMATICS COURSE

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Promoting student wellbeing in mathematics may be one way to tackle long-standing equity issues in tertiary mathematics education. To expand our understanding of wellbeing in domain-specific settings, this research paper presents findings from a pilot study examining the relationship between wellbeing and parental educational attainment in 140 predominantly first- and second-year college students in the United States taking an introductory statistics course. Findings suggest that first-generation college math students experience a greater sense of engagement, meaning, and—most notably—accomplishment in their math classes than students from higher educational backgrounds. This study frames student wellbeing in mathematics today as an issue that is highly relevant for universities and math departments in the long run, offering a way to measure the wellbeing of students via a five-dimensional operational model.

INTRODUCTION

Examining the intersection of student wellbeing and parental educational attainment in college mathematics courses prompts consideration of the link between social class and wellbeing. Previous research by Dougall et al. (2021) indicates that students from elevated social classes experience higher wellbeing in college compared to their counterparts from lower social strata. In the context of lower social classes, mathematics curricula often stand out as seemingly "devoid of any particular rationale" (Gates, 2019, p. 44). In fact, in the United States, even the majority of parents seem to agree that the content of higher mathematics curricula seems devoid of relevance to the lives of many students today (Blad, 2023). However, the global significance of math education underscores its profound importance (Burdman, 2018; Hill & Seah, 2022). In today's landscape, high mathematics performance not only influences college admissions decisions (Anderson & Burdman, 2021) it also shapes future earning potential (Carnevale et al., 2011). Mathematics thus serves as both a gateway and gatekeeper to social mobility.

For many students, navigating the world of mathematics becomes akin to a pressure cooker system, where academic performance acts as both doorways and gatekeepers. This intense environment, particularly for students aspiring to upward social mobility, can magnify stress, anxiety, and lead to misconceptions about the field by the time they enter college. Recognizing this narrative in the context of mathematics education is paramount, especially when designing postsecondary mathematics policies (Burdman, 2018).

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25–31). PME.

Recent studies emphasize the intertwined nature of student wellbeing and mathematics achievement (Yao et al., 2018). Moreover, social class exhibits a significant correlation with students' learning outcomes in mathematics (Gates, 2019). Despite these insights, scant research delves into understanding the wellbeing of mathematics students at the postsecondary level (c.f., Almora Rios, 2023). Furthermore, the literature lacks exploration connecting social class to anti-deficit notions of wellbeing at the domain-specific level.

In response to these gaps, this paper presents findings from a pilot study examining the relationship between social class and anti-deficit wellbeing in the context of the tertiary mathematics classroom. Concepts for wellbeing and social class are briefly discussed, as well as the construction of the survey instrument. Univariate ANOVA is used on the datum to extrapolate findings from student responses. Finally, further research directions are discussed.

THEORETICAL FRAMEWORK

Wellbeing

Founded on Aristotelian ideology, 'wellbeing' is defined conceptually as "the unfolding of natural, fixed, or innate potentialities….[or] a right, optimal, or perfect functioning that is teleologically fixed as the realization of innate patterns of growth" (Nafstad, 2015, p. 13). It is a combination of feeling good, functioning well, and contributing positively to a community (Chaves, 2021). Operational definitions usually include a hedonic (i.e., *subjective* wellbeing) and eudaemonic component (i.e., living in accordance to one's values). One framework pertinent to educational settings is Seligman's PERMA model (Seligman, 2011). The PERMA model operationalizes wellbeing through five dimensions: Positive emotions, Engagement, Relationships, Meaning, and Accomplishment. Because recent work has also localized wellbeing as a context-dependent construct (Alexandrova, 2017; Hill et al., 2021) this paper poses wellbeing as a *domain-specific* construct. In short: a student's experience of wellbeing in, say, a math class may be operationally or conceptually different than in the general sense (see Hill et al., 2021). Conceptions thus importantly range across settings, cultures, populations and time.

Social Class

Social class comprises of an economic and social ordering, with one's social order reacting to and being pre-conditioned by their economic order (Weber et al., 2009). While social class begets the distribution of status, honor, and prestige, it is "the most defining characteristic that influences attainment, achievement, and engagement in schooling" (Gates, 2019, p. 42) underscoring—importantly—a plethora of educational outcomes. Though inherently difficult to measure, parental educational attainment is often used as an estimate for social class when other data (such as income) may be hard to come by (Dougall et al., 2021). As such, this paper uses students' highest parental educational attainment levels as a proxy for the complex measure of social class.

RESEARCH GOAL

This project answers the following question: does parental educational attainment (i.e., an estimator for social class) affect students' experience of wellbeing in a first-year collegiate math course traditionally for non-math majors?

Participants

Participants comprised of 140 students (67 female, 70 male, 3 gender non-conforming) from a one-semester introductory course to probability, statistical reasoning, and linear models at the University of Montana during the Fall 2022 semester. Around 89% of the sampling pool identified as White. Other ethnicities included Asian or Pacific Islander (2.7%), Hispanic or Latino (2.7%), Black or African American (1.38%), and Native American or American Indian (.69%). Sample included 31 students with both parents holding a high school or middle school degree as their highest degree, 59 students with at least one parent holding a college degree as their highest degree, and 50 students with both parents holding a graduate degree. Students were majority first- and second-year students (105 first-year, 21 second-year, 11 third-year, 3 fourth year or higher) majoring in flavors of social and business sciences.

Methods

Students were tasked to complete an anonymous 23-item Likert-type survey instrument on their course webpage. Extra-credit points were awarded for completing the questionnaire in a two-week period during the Fall 2022 semester. The following student groups were highlighted in this analysis: students whose parents' highest educational attainment was secondary school (i.e., middle school or high school; a.k.a. first-generation college students), students with at least one parent holding a college degree, and students with at least one parent holding a graduate degree. Univariate ANOVA was used with post-hoc Tukey tests to assess statistically significant effects in students' ratings along each of Seligman's five PERMA dimensions of wellbeing, and Butler & Kern's (2016) two additional dimensions of Negative Emotions and Health.

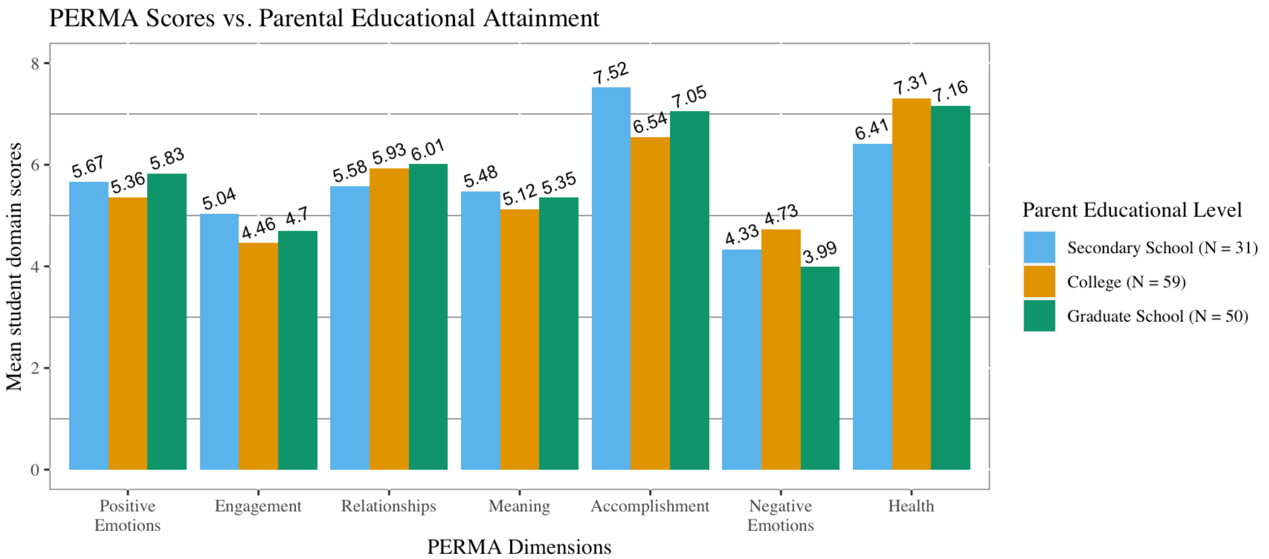
Following similarly the analysis of Butler and Kern (2016), seven "domain scores" were created for each student. Domain scores were calculated by averaging students' scores along three scale items corresponding to each of Seligman's (2011) five PERMA dimensions and Butler and Kern's (2016) two additional dimensions of Negative Emotions and Health. For this study, domain score differences greater than 0.4 on a Likert scale between the three groups of parental educational attainment were considered practically significant.

Tools

The questionnaire was based on the Workplace PERMA Profiler, a 23-item psychological scale measuring an individual's 'wellbeing score' along dimensions of positive emotions, engagement, positive relationships, meaning, accomplishment, negative emotions, health, loneliness, and happiness within the context of a workplace environment (Kern, 2014). The Workplace PERMA Profiler is a free-to-use, open-access instrument accessible on the author's website. The instrument was adapted in this study to fit the context of the workplace that is the collegiate math class. The adapted Profiler adds context-specific phrases to questions from the Workplace PERMA Profiler to measure the wellbeing of students in a collegiate math course along the five PERMA dimensions and the two dimensions of Negative Emotions and Health listed above. For example: How often do you feel you are making progress towards accomplishing your work-related goals? from the Workplace PERMA Profiler (Kern, 2014) becomes How often do you feel you are making progress towards accomplishing your work-related goals as a student in a math classroom? on the adapted Profiler. Cronbach's alpha showed acceptable internal consistency (alpha = .782) for the seven domains measured.

FINDINGS

Mean ratings for the PERMA, Negative Emotions, and Health dimensions can be seen



in Figure 1.

Figure 1: Mean wellbeing scores across parental educational levels. Statistically significant effects in experience of Accomplishment ($F(2, 137) = 3.1$, $p < .04$).

Students with parents holding graduate degrees (green; see Figure 1) experience higher levels of positive emotions and positive relationships than do first-generation college students (blue), as well as lower levels of negative emotions. The higher scores in the health domain by the graduate degree and college degree-attaining group also mirrors work on socioeconomic conditions to signs of physical health concerns (Herd et al., 2007). Univariate ANOVA also reveals parental educational level as having a statistically significant effect on students' experiences of *accomplishment* in college math classes $F(2, 137) = 3.1$, $p < .04$. Tukey post-hoc tests place this difference between the 'College' (orange) group and the 'Secondary School' (blue) group ($t = 2.513$, $p < 0.0346$). Interestingly, other domains did not show signs of significant differences at this time, an effect most likely due to limited access to larger sample sizes of first-generation college students in the study.

Emergent 'staircase patterns' within the Profiler domains may also suggest relationships between parental educational attainment and student wellbeing. For instance, in the Relationships domain, parents' educational levels may play a helping role in influencing students' experience of feeling valued by others and connected to others in their math classes, as seen in the rising Relationships scores alongside the rising parental educational levels.

DISCUSSION AND CONCLUSION

This pilot study aims to assess the wellbeing of students in higher mathematics education and introduces measurement instrumentation for this purpose. The findings reveal that students from higher educational backgrounds feel more connected to their math peers and instructors, potentially experience less negative emotions from their math work (e.g., frustration, anxiety), and report greater physical health overall (affecting sleep, stress, and retention) than first-generation students. On the other hand, first-generation students feel greater engagement in their math coursework, a greater sense of meaning, and—most notably—a significant sense of accomplishment in their work, compared to students with parents holding a college degree. Social class, measured via parental educational attainment, emerges as a pivotal factor influencing students' perceptions of accomplishment in college math classes.

While some argue that the distinct experiences of first-generation college students underscore the need for targeted interventions and support mechanisms tailored to their unique challenges within the mathematics education landscape, an anti-deficit perspective suggests that the responsibility for addressing these challenges should lie with the institutions that are supposed to nurture these students.

To thoroughly understand the underlying reasons for the differences in mean domain scores (see Figure 1), a more qualitatively oriented project involving interviews with students from each group may be necessary. Acknowledging the limitations of this study, particularly the restricted sample size of first-generation college students, is also essential. Greater sampling methods, perhaps across university systems, may contribute to statistically significant differences in other domains. Future research should also prioritize including a more racially diverse student sample to provide a more comprehensive understanding of the experiences and challenges of college students in a first-year math course, as well as include grades or measures of retention to help reify the wellbeing construct.

Mathematics is a discipline whose practice is fully capable of promoting a flourishing life in learners (Su & Jackson, 2020). This aspect of mathematics practice—just as much as the opportunity it inspires—is deserving of equitable distribution across student populations. This study serves as a starting point for further research—especially for mixed methods analyses—encouraging exploration to a more nuanced picture of students' wellbeing and its connection to social class in college mathematics education. Given the limited use of the PERMA Profiler on student populations outside of Australia (c.f., Hill et al., 2022; Hill & Seah, 2022; Almora Rios, 2023), introducing wellbeing measurement to the international mathematics community will contribute to fostering a perceived lifelong learning culture in mathematics. The current lack of research on student wellbeing in mathematics classes, especially in the U.S., highlights the necessity of structuring systems (such as mathematics departments) to promote the wellbeing of students. As such, this project aims to improve educational equity in STEM fields—an area that requires much further attention, especially at the postsecondary level.

Acknowledgements

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. 2034835.

References

Almora Rios, M. (2023). "Should wellbeing be the same as perfection?" A case study on student wellbeing in higher mathematics education. *Graduate Student Theses, Dissertations, & Professional Papers.* 12037.

Alexandrova, A. (2017). *A philosophy for the science of well-being*. Oxford University Press.

Anderson, V., & Burdman, P. (2022). *The calculus of college admissions: How policy, practice, and perceptions limit equity in high school mathematics*. Just Equations.

Blad, E. (2023). Parents, teachers, agree: Math matters, but schools must make it relevant. *EdWeek*. https://www.edweek.org/teaching-learning/parents-teachers-agree-math-matters-but-schools-must-make-it-relevant/2023/04

Burdman, P. (2018). *The mathematics of opportunity*. Just Equations.

Butler, J., & Kern, M. L. (2016). The PERMA Profiler: A brief multidimensional measure of flourishing. *International Journal of Wellbeing*, *6*(3), 1-48. https://doi.org/10.5502/ijw.v6i3.526

Carnevale, A. P., Smith, N., & Melton, M. (2011). *STEM: Science, technology, engineering, mathematics*. Georgetown University Center on Education and the Workforce.

Chaves, C. (2021). Wellbeing and flourishing. In: M. L. Kern, & M. L. Wehmeyer (Eds.), *The Palgrave handbook of positive education* (pp. 273-295). Palgrave Macmillan, Cham. https://doi.org/10.1007/978-3-030-64537-3_11

Dougall, I., Weick, M., & Vasiljevic, M. (2021). Social class and wellbeing among staff and students in higher education settings: Mapping the problem and exploring underlying mechanisms. *Journal of Applied Social Psychology*, *51*, 965-986.

Gates, P. (2019). Why the (social) class you are in still counts. In: C. Xenofontos (Ed.), *Equity in mathematics education: Addressing a changing world* (pp. 41-64). Information Age Publishing Inc.

Herd, P., Goesling, B., & House, J. S. (2007). Socioeconomic position and health: The differential effects of education versus income on the onset versus progression of health problems. *Journal of health and social behavior*, *48*(3), 223–238. https://doi.org/10.1177/002214650704800302

Hill, J. L., Bowmar, A., & Hunter, J. (2022). A preliminary study exploring the mathematical wellbeing of grade 3 to 8 students in New Zealand. In: C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 387-394). PME.

Hill, J. L., Kern, M. L., Seah, W. T., & van Driel, J. (2021). Feeling good and functioning well in mathematics education: Exploring students' conceptions of mathematical well-being and values. *ECNU Review of Education*, *4*(2), 349-375. https://doi.org/10.1177/2096531120928084

Hill, J. L., & Seah, W. T. (2022). Student values and wellbeing in mathematics education: Perspectives of Chinese primary students. *ZDM - Mathematics Education*, *55*, 385-398. https://doi.org/10.1007/s11858-022-01418-7

Kern, M. L. (2014). *The workplace PERMA profiler*. Retrieved August 30, 2022 from https://www.peggykern.org/uploads/5/6/6/7/56678211/workplace_perma_profiler_102014.pdf

Nafstad, H. E. (2015). Historical, philosophical, and epistemological perspectives. In: S. Joseph (Ed.), *Positive psychology in practice: Promoting human flourishing in work, health, education, and everyday life* (2nd ed., pp. 9-29). https://doi.org/10.1002/9781118996874.ch2

Seligman, M. E. P. (2011). *Flourish*. New York, NY: Simon & Schuster.

Su, F., & Jackson, C. (2020). *Mathematics for human flourishing*. Yale University Press.

Yao, Y., Kong, Q., & Cai, J. (2018). Investigating elementary and middle school students' subjective well-being and mathematical performance in shanghai. *International Journal of Science and Mathematics Education, 16*, 107-127. https://doi.org/10.1007/s10763-017-9827-1

STUDENTS' CONCEPTIONS ABOUT MATHEMATICS FOR CLIMATE CHANGE AND RELATED ISSUES

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In recent PME and ICMI conferences, a need for curriculum innovation that takes into account the role of mathematics in understanding and contrasting climate change and related issues has been stressed by prominent scholars, taking a rather cognitive stand. In this paper, we focus on the affective side of the phenomenon, arguing that the students' conceptions both about mathematics and about climate change and related issues need to be taken into consideration in order to make such an innovation effective. Hence, we report and analyse the narratives that a small sample of students enrolled in an Environmental Sciences program produced during the activity of writing a letter to a fictitious class of students living in the future describing how mathematics has helped humans to survive in the next 200 years.

INTRODUCTION AND THEORETICAL BACKGROUND

Mathematics, according to Coles (2023), plays a central role in dealing with, yet responding to, issues like climate change, population growth, pollution, resource scarcity and wastefulness, but the mathematics that is taught at school is scarcely (if not at all) connected to these ideas (Coles, 2023). Coles (2023) shows examples of curriculum innovation for mathematics more connected to these socio-ecological issues that emerge in the new climatic regime. He proposes changes of the content that students should learn in mathematics classes. In this paper, we propose to add reflections on the role of students' conceptions, both about mathematics and about socio-ecological issues, being the latter a relevant yet unexplored element that influences the learning of mathematics (Coles, 2023). According to many studies (e.g., Sumpter, 2013), the ways students engage with mathematics depend on their different affective disposition. A variety of researches focused on how mathematics is perceived and dealt with, such as the different motivation students' express (Nyman & Sumpter, 2019), or how different expectations function as a mediator for various choices students make when solving mathematical tasks (Sumpter, 2013). Among all the mentioned affective dimensions, beliefs received special attention in research. According to Furinghetti and Pehkonen (2002), beliefs are the conclusions that an individual draws from their perceptions and experiences in the world around them. Beliefs can be understood as subjective knowledge: they are propositions about a certain topic that are regarded as true (Philipp, 2007). Being continuously subject to new experiences, beliefs can change and new beliefs can be adopted (Furinghetti & Pehkonen, 2002). When a new belief emerges, it never comes in isolation from other beliefs, but becomes part of, what has been called, an individual's belief system. According to Green (1971), in fact, beliefs tend to form clusters, as they "come always in sets or groups, never in

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 32–39). PME.

complete independence of one another" (Green, 1971, p. 41). These clusters form a system, which is organised according to the quasi-logical relations between the beliefs and the psychological strengths with which each belief is held (Green, 1971). Belief clusters are, thus, almost coherent families of beliefs across multiple contexts: for example, beliefs about the nature of mathematics and about its learning tend to cluster in a quite coherent way, for a student. This has probably led Furinghetti and Pehkonen (2002) to conclude that "an individual's conception of mathematics [is] a set of certain beliefs" (p. 41), namely to understand conceptions as clusters of beliefs. Liljedahl (2018) further prompts the research field to consider beliefs not as operating as singular entities, but in synergy with emotions and attitudes, to form what he called an affective system. Several researchers stress how motivation, emotions, and beliefs are intertwined, with each other (e.g., Liljedahl, 2018) or internally such as different types of motivation being combined in one statement (Nyman & Sumpter, 2019), or such as attitudes being conceived as an amalgam of emotional disposition, perceived competence and view of mathematics (Di Martino & Zan, 2011). For this reason, in this paper we decided to use the construct "conceptions", which is meant as an umbrella concept, namely: "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007, p.259). Hence, conceptions may have both affective and cognitive dimensions and serve the purpose of capturing students' ideas and dispositions (Philipp, 2007).

The aim of the paper is to showcase a pilot, small-scale study on students' conceptions about mathematics and about climate change and related issues. To that end, an activity consisting of writing a letter to fictitious future students living 200 years ahead to show how sciences and mathematics contribute to the survival of life has been carried out. This kind of activity is called speculative storytelling (Helliwell & Ng, 2022): the arts and humanities can provide opportunities to engage with socio-ecological issues and to alter attitudes and behaviours in ways that formal scientific approaches on their own do not. It has been argued (Helliwell & Ng, 2022) that teaching and learning approaches involving a variety of art forms and aesthetic elements have qualities that could develop education, especially with respect to climate change. This kind of issues is commonly perceived as distant and abstract, but arts and humanities can contribute to make it closer and concrete for students (Helliwell & Ng, 2022). Issues like climate change are also overwhelming and difficult to grasp, but arts and humanities can help express negative emotions (Helliwell & Ng, 2022). Helliwell and Ng (2022) further maintain that to work in this way entails drawing on multiple forms of knowing (i.e. cognitive, sensible, somatic, affective). Specific to speculative storytelling, Helliwell and Ng (2022) utilised it as a curriculum innovation. The researchers used speculative storytelling primarily as a way of engaging the participants in sharing their ideas and collaborating around the topic of sustainable futures in the mathematics classroom. The researchers also recognised a potential for speculative fiction as a pedagogical tool for prompting students' imagination, deemed as a way to conceptualise alternatives to what is taking place and, thus to realise that certain facts are contingencies and not necessities. According to Helliwell and Ng (2022), it is this kind of imagination that enables a suspending and letting go of taken-for-granted ways of being to contemplate more just and equitable futures.

In our research, we apply these ideas to engage students in speculative storytelling about how mathematics can help surviving for the next 200 years. This activity has a double goal: it allows conceptions about mathematics, brought from school, to emerge, and it prompts students to imagine other roles of the discipline beyond the boundaries of what was taught to them during the school years. Thus, the research question we aim at answering is: can the conceptions that emerge in an activity of this sort contribute to understand how students approach not only mathematics, but also the mathematical activities centred on socio-ecological issues?

METHODS

The participants to the study are 32 students enrolled in the first year of an undergraduate program in Environmental Sciences, in an Italian University. They are 10 females and 22 males aged 20 years, with an exception of two students who are 28 years old. At the time of data collection, they were attending the first lecture of the mathematics course, led by the author. The sample represents 70% of all the students enrolled (other 12 students usually did not show up during the classes).

With respect to data collection, it is well acknowledged that much of the studies on affective aspects have been conducted through narratives such as essays, diaries, questionnaires with open questions and interviews (e.g., Kaasila, 2007; Di Martino & Zan, 2011). In line with the method of narrative data collection (Kaasila, 2007), the students of our sample were asked to answer an open prompt, that is:

> We got a letter from the future: the people who live on Earth in 200 years wrote to us. They say that on Earth there is life, it is possible to breathe fresh air, to drink water and there are the conditions for thinking about the future. They ask us, however, to tell them how we made it possible and which was the role played by mathematics (and sciences). Answer to them, individually.

The students were given 20 minutes to reply, and data were collected anonymously. Each student had been assigned a label, like S1 for student 1. The goal of the narrative approach is to get the respondents to tell stories about things that are important to them, feeling free to express their conceptions, reporting the aspects that they consider central in their own experience (Kaasila, 2007; Di Martino & Zan, 2011). Moreover, with open prompts, respondents are not forced to align their opinion on a ready-made list chosen by the researcher (Di Martino & Zan, 2011).

The collected narratives were analysed according to holistic and categorical approaches (Lieblich, Tuval-Mashiach & Zilber, 1998). In a holistic approach, the narrative is analysed as a whole, and the focus is on the overarching themes that emerge from whole responses, instead of focusing on specific terms or concepts that are expressed in a specific text. For example, the narratives with respect to one’s relationship with mathematics can refer to themes which span from the positive feelings during primary school days to the anxiety before the exams. These themes, emerged from the data and not created in advance by the researcher, are considered for holistic analysis and grouping. In a categorical approach, in each narrative, sections or even single words are taken into account (Lieblich *et al.*, 1998) and then classified by the researcher through semantically identifying expressions that refer to a same category (also categories emerge from the data). Elaborating on the previous example, some students may mention the pleasure of working with geometrical figures, thus their narratives (sections or words) are grouped by the researcher in a specific category (e.g., "GF"), others may recall counting games, contributing to a different category (e.g.,"CG"), and so on. In this way, the overarching theme of positive feelings with respect to mathematics at primary level is specified in categories "GF", "CG" and so on. A narrative from a single student can contain expressions that belong to different themes and categories, and some categories might not belong to a unique theme. Moreover, some categories, which emerge from sections and words, might not be associated to any theme. The combination of a holistic and a categorical approach, allows for a deeper and differentiated understanding of the narratives (Kaasila, 2007) firstly focusing on the general, overarching themes that emerge across the narratives, then going into details focusing on the categories. Accordingly, it is appropriate to apply the classification made by Lieblich *et al.* (1998) as "an analytical bridge: the ultimate purpose can be to integrate the approaches into a whole" (Kaasila, 2007, p.5). This method catches and operationalizes, in our view, the idea developed in our theoretical framework that conceptions form a system (Philipp, 2007; Liljedahl, 2018).

DATA ANALYSIS

The holistic approach allows to identify two general themes that emerge from the narratives and that are recurrent across several narratives: *progress* and *role(s) of mathematics*. The first theme emerges in 17 narratives: the students use expressions like: progress, development, increase. They mention technological progress, progress of knowledge and culture, scientific progress, as means that would make life possible on Earth in 200 years. In students' narratives, there is a trust in the progress as a way to mitigate and contrast the existing trend. The second holistic theme, which can be found in 15 narratives, concerns the *roles of mathematics*. In this case, mathematics is not only mentioned, nut the possibilities offered by the discipline to save the world are specified. The roles of mathematics that mostly emerge from the narratives are: computation; data analysis to get the sense of the extent of pollution, hanger, resource use, energy, wastefulness; problem solving; estimation of risk; modeling. Taking on a systematic stance on students' conceptions, the holistic analysis allows us to infer that these two main themes span across students' narratives.

Categories

Afterwards, with the categorical approach we identify words in the narratives possibly related to the two holistic themes. With respect to the *progress* theme, we labelled the words and statements used by the participants in 4 categories, while to the holistic theme of *roles of mathematics* other 5 categories have been attached. Furthermore, we identify other 3 categories that do not relate to any holistic theme. We also note that, with respect to gender, we observed no particular difference in the ways females and males express their ideas, nor a predominance of certain themes or categories in either gender group.

The holistic theme "*progress*" is associate to 4 categories: technological progress (T, 9 students), mathematics as key for progress (K, 8 students), discovery (D, 6 students) and acquisition of new knowledge (A, 4 students). Examples are:

> New technologies have been implemented and they have improved our lifestyle, reducing the risk of floods and earthquakes and limiting their gravity and flow (S12, category T).

> As regards the aspect of planetary conservation, through the discovery of new chemical elements and the improvement of existing ones, new substances have been created capable of neutralising all the polluting effects of materials, such as plastic, to encourage growth and ecosystem development. In the context of other disciplines, it has been possible to solve world hunger by creating fast and efficient means of transport that reach all points of the Earth, powered by solar energy. Through the in-depth study of space launches on the Moon, it was possible to reach Mars and make the most of its resources (S6, T).

> Everything is related to knowledge that has increased constantly. The fact that in 200 years life is possible on Earth prompts me to think that this trend did not decrease but it has increased (S1, A).

> A slow improvement to the social, economical and political situation has taken place thanks to an incessant development of science and mathematics (S5, A).

> Mathematics and science have been the keys of the progress since ancient times, and continue to be (S2, K).

> Mathematics and science are essential to a progress that is aimed at safeguarding the well-being of life and that of the planet (S23, K).

> Also in the past, Pitagora, Euclid, Gauss, Newton, Einstein, Galilei are among the mathematicians and scientists who changed the world (S2, D).

> Mathematics has developed and has been able to find answers as long as the questions become more complicated (S26, D).

The first two excerpts focus on technological progress in general (S12) and on specific technological innovations in particular (S6) and have been identified within the category T. The third and the fourth excerpts focus on the progress of knowledge. Student S23's statement has been related to category K, while the last two statements are examples for category D, which includes also change. One can notice that the word "progress" is either explicitly mentioned in these excerpts, or it is evoked by expressions that relate to it, while mathematics is not always mentioned explicitly. In the other 19 statements (of the 27 in total) that are not reported here, only 6 explicitly mention mathematics, and this is interpreted as if the students are not always aware of the importance of the discipline for technological progress. We stress that those who mention mathematics under this theme, they mention it in a general way, detached from possible practical implications and uses of the discipline towards progress and innovation (see in the last four examples reported): a specific role for mathematics is not described, nor how it contributed concretely to progress).

Within holistic theme of the *roles of mathematics* we found five categories: solving problems (S, 8 students), analysing data (DA, 8 students), estimating probability of impact (P, 5 students), explaining (E, 2 students), modelling (M, 3 students). Examples are:

> Mathematics has provided us with solutions to many problems (S28, S).
>
> I give you an example: we know that global warming is one of the most important problems nowadays and thanks to mathematics we have been able to locate the problem, search for a solution, apply the solution and monitor if this solution works (S4, S).
>
> Thanks to countless surveys, studies and research done on man and regarding his habits and vices, and on nature, we have managed, albeit slowly, to change the fate of our planet (S5, DA).
>
> Researchers and data analysts collect data about all the issues (S18, DA).
>
> Without data analysis, without a tool to control the data, we would not be aware of the gravity of certain situations and, thus, we would not do enough to improve (S14, DA).
>
> Mathematics and science have contributed to give an explanation to all phenomena that were inexplicable (S2, E).
>
> Through precise and complex mathematical computations we will be able to optimise the resources and to use them in various contexts (S23, M).
>
> Mathematics has allowed to create models that favour the social system, which was precarious in the beginning (S6, M).

In these example, mathematics’s roles are detailed, as well as in the other narratives not reported. Mathematics is associated mostly to computations and handling of data, and more rarely to modeling and predicting.

Other three categories not related to a specific holistic theme emerged: collaboration among disciplines (C, 8 students); the relationship between theory and practice (TP, 5 students); the role of education (ED, 4 students).

> The interaction among disciplines has led to enormous steps forward (S1, C).
>
> Mathematics allows for the solution of real problems through a theoretical approach, science is a discipline that that has numerous applications and concerns the pragmatic side of phenomena, giving explanations through observations and experiments (S7, TP).
>
> Thanks to people like us, we will be able to educate people to a more sustainable way of living (S18, ED).

One can notice that also in these examples, mathematics is mentioned in limited cases, but when it emerges (e.g., in S7), it is compared to sciences and a specific role is recognised to the discipline. Finally, one student admits that he never thought about the role of mathematics in tackling these issues and has no idea (S22). No category has been assigned to it.

Figure 1 summaries our holistic-categorical approach analysis: the two holistic themes (green circles) are *progress* and *roles of mathematics* and are linked with categories (orange circle), while the others are disconnected. For our purpose, the two identified holistic themes can be seen as a first classification of conceptions about the topic under analysis, while the categories represent a specification on those conceptions. Of course, these themes are not surprising, because it is well acknowledged the importance of progress to mitigate the effects of climate change and to adapt to the new regime, and because the task explicitly asked about mathematics. In a sense, it is not surprising for us that students' conceptions can be grouped under these themes, as conceptions encompass beliefs, meanings, concepts, propositions, rules, mental images, and preferences (Philipp, 2007) about climate change and related issues in our study. Moreover, the four conceptions linked to progress, for example, allow us to better specify the conceptions students have on progress, which focus on technology, development of knowledge, discovery and on mathematics as a basis for it. With respect to mathematics, we can conclude that students' conceptions of the discipline as a tool to solve problems and deal with data are prevalent.

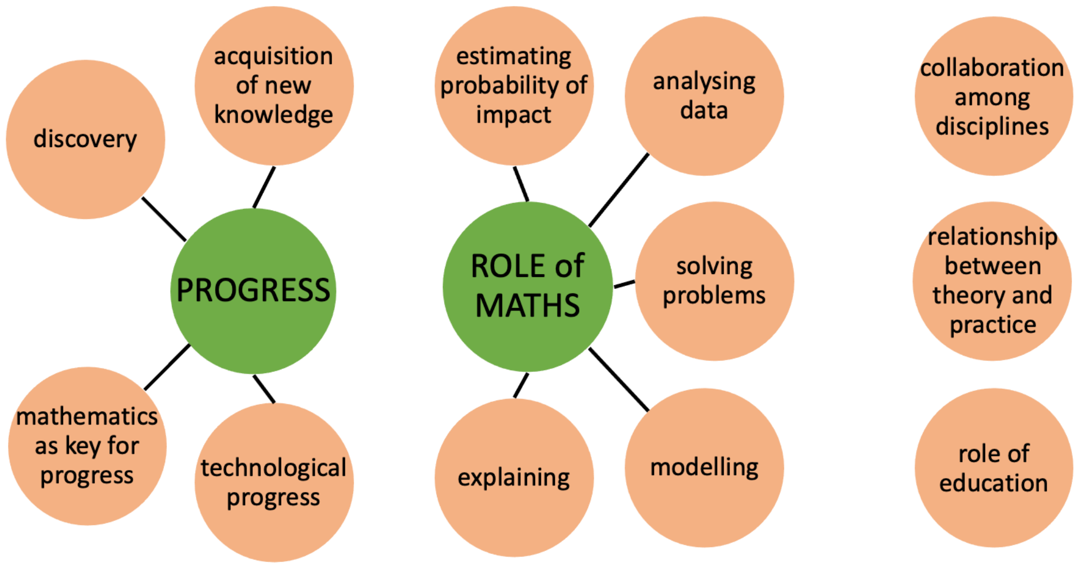

Figure 1: The holistic themes (green) and the categories (orange) that emerge from the data analysis. A line for categories that can be connected to a theme is drawn.

DISCUSSION AND CONCLUSION

We explored conceptions about role of mathematics in contributing to contrast and mitigate issues like climate change and to make life possible in the next 200 years. We identified two holistic themes and found 12 categories for conceptions. We noticed that the students do not always mention mathematics explicitly when talking about progress and the role of education. In our interpretation, and according to the theoretical framework, this could be due to the fact that mathematics at school is taught without application to the world (Coles, 2023): this means that even in activities that leave the imagination free and that are specifically designed to prompt such an imagination (Helliwell & Ng, 2022), students seem unable to see a role for mathematics. In other words, in this research possible futures are imagined by the students (Helliwell & Ng, 2022), leaving the mathematics relatively aside, or with a very vague role. In the conceptions associated with the roles that mathematics takes on, the discipline is central but emerges often as calculations on data, rarely as modeling or a tool for making predictions. This reflects, in our interpretation, the kind of mathematics that students learn and do at school, namely rote exercises, computation and algorithms. As a conclusion, we argue that in innovating mathematical curriculum, it is necessary to monitor also if and how the students' conceptions change to make it possible for them to consider new ways of seeing mathematics, especially as a discipline that contributes concretely to progress.

References

Coles, A. (2023). Teaching in the new climatic regime: steps to a socio-ecology of mathematics education. In: M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-34). Haifa.

Di Martino, P. & Zan, R. (2011). Attitude towards mathematics: a bridge between beliefs and emotions. *ZDM 43*(4), 471-482. http://dx.doi.org/10.1007/s11858-011-0309-6

Furinghetti, F. & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In: G. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 39-57). Dordrecht/Boston/London: Kluwer Academic Publishers.

Green, T. F. (1971). *The activities of teaching*. Tokyo: McGraw-Hill Kogakusha

Helliwell, T & Ng, O-L. (2022). Imagining possibilities: innovating mathematics (teacher) education for sustainable futures. *Research in Mathematics Education, 24*(2), 128-149.

Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM, 39*(3), 205-213. https://doi.org/10.1007/s11858-007-0023-6

Lieblich, A., Tuval-Mashiach, R., & Zilber, T. (1998). *Narrative Research. Reading, Analysis and Interpretation.* (Vol. 47) Sage. http://dx.doi.org/10.4135/9781412985253

Liljedahl, P. (2018). Affect as a system: The case of Sara. In: B. Rott, G. Törner, J. Peters-Dasdemir, A. Möller, & Safrudiannur (Eds.), *Views and beliefs in mathematics education* (pp. 21-32). Springer. https://doi.org/10.1007/978-3-030-01273-1_3

Nyman, M. & Sumpter, L. (2019). The issue of 'proudliness': Grade 2 and Grade 5 students' motivation towards mathematics. *LUMAT: International Journal on Math, Science and Technology Education, 7*(2), 80-96. https://doi.org/10.31129/ LUMAT.7.2.331

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. *Second handbook of research on mathematics teaching and learning, 1*, 257-315.

Sumpter, L. (2013). Themes and interplay of beliefs in mathematical reasoning. *International Journal of Science and Mathematics Education, 11*(5), 1115-1135.

MATHEMATICS TEACHER EDUCATORS' EXPERTISE BASED ON PEDAGOGICAL COMMUNICATION

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This study aimed to understand features of expertise of Mathematics Teacher Educators (MTEs) based on their pedagogical communication in the form of academic booklets to support preservice teachers' learning. The booklets, authored by experienced Brazilian MTEs, were analysed using grounded theory methods. The findings offer insights of MTEs' expertise in terms of three pedagogical contexts (mathematics, teaching mathematics, and academic research) and bridge-building skills linking these contexts. The study offers a basis to enhance our understanding and conduct future research on MTEs' expertise.

INTRODUCTION

Research on the Mathematics Teacher Educator (MTE) is in its infancy (Beswick & Goos, 2018) compared to research on mathematics teachers that have had significant attention over the last few decades. The need to research Mathematics Teacher Educators (MTEs) is essential to understand how they could most effectively prepare and further develop mathematics teachers for a rapidly changing world (Chapman et al., 2022). Recent studies on MTEs have focused on those formally responsible for the professional development of mathematics teachers (Chapman, 2021; Coura & Passos, 2021; Martignone et al., 2022; Masingila et al., 2018). These studies provide insights into MTEs' mathematical knowledge and practice, but as Chapman (2021) argued, there needs to be consideration of alternative or expanded ways of researching and understanding the MTE in relation to their work with educating teachers. In this paper, we offer a possible way of doing this through a study that investigated MTEs' expertise, which is interpreted broadly as an amalgam of knowledge, social participation, and communication to teach mathematics teachers. This study specifically focused on university-based MTEs and their expertise in preparing preservice teachers through direct mathematics-related teacher education courses or strategies. The aim was to identify key features of their expertise based on their pedagogical communication to future teachers and new MTEs to further understand their expertise. Following, we present related literature and theoretical perspectives, the methodology, the results, and discussion of the findings.

RELATED LITERATURE AND THEORETICAL PERSPECTIVES

Research on MTEs suggests specific ways of conceptualizing or understanding their knowledge, practice, learning, and development (Beswick & Chapman, 2020; Goos & Beswick, 2021). These ways tend to build on those used in research on mathematics teachers. For example, research on MTEs often compares the specific mathematics

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 40–47). PME.

knowledge of MTEs with that of mathematics teachers. Acknowledging that teachers possess distinct mathematical knowledge, research has similarly recognized that MTEs have a specific knowledge base (Chapman, 2021). This has implied efforts to expand theoretical frameworks initially developed for teachers' mathematical knowledge to better understand MTEs' knowledge (Chapman, 2021).

Efforts to conceptualize MTE knowledge have led to the development of frameworks like Mathematical Knowledge for Teaching Teachers (MKTT), inspired by the Mathematical Knowledge for Teachers models. Researchers like Masingila et al.(2018) and Superfine et al. (2020) have attempted to describe MKTT in various domains. Martignone et al. (2022) expanded the Mathematics Teachers' Specialised Knowledge (MTSK) model to create the Mathematics Teacher Educators' Specialised Knowledge (MTESK) framework, which includes knowledge of teaching and learning mathematics for students and teachers, as well as research knowledge in mathematics education.

Beswick and Chapman (2015) raised the question of the distinctiveness of MTE knowledge, suggesting it might be a form of meta-knowledge. Subsequent studies (e.g., Beswick & Goos, 2018) have recognised this meta-knowledge as part of MTE knowledge. Beswick and Goos (2018) emphasised that MTE knowledge includes understanding how teachers learn and develop competence. However, as Chapman (2021) noted, the field still engages with diverse theoretical models and their adaptations for MTEs from models used for mathematics teachers. Moreover, the traditional focus on "knowledge" within the teacher thinking paradigm faces challenges from social, situated, and communicational perspectives. Thus, there is an ongoing need to explore beyond existing models. Helliwell and Chorney (2022) suggest a focus on MTEs' expertise as encompassing more than individual knowledge, incorporating material and social factors. Accordingly, our study examines MTEs through the lenses of expertise and pedagogical communication.

MTEs' expertise is interpreted broadly in relation to practice, competence, skill, and knowledge to teach mathematics teachers. We conceptualise MTEs' expertise as an amalgam of knowledge, social participation, and communication, reflecting the specific know-how of MTEs in their role as educators. It encompasses how MTEs anticipate, communicate, and facilitate pedagogical interactions with teachers. Thus, similar to Helliwell and Chorney's (2022) position, we view MTEs' expertise not as a separate entity from how they pedagogically carry out and communicate their work. Their expertise includes how they organise and carry out oral and written pedagogical communication with prospective or in-service teachers to support their learning.

The construct of pedagogical communication can be seen in terms of the relationship between the educator and the learner, which occurs through verbal, written, visual, or gestural forms, with the purpose of constituting the message considered legitimate (Bernstein, 2000). From this perspective, communication, which is pedagogical because it has an educational purpose, involves the specialization, selection, sequencing, pacing, and criteria regarding the legitimate knowledge to be taught (Bernstein, 2020). In this exploratory study of MTEs' expertise, we adapted these notions for researching experienced MTEs who developed written materials to support preservice teachers' learning of mathematics for teaching and new/inexperienced MTEs' learning to teach preservice teachers (PTs). This form of communication represents the MTEs' expertise that they are sharing with PTs and new MTEs. Thus, it offers a means for us to explore the expertise of the MTEs. This combination of MTEs' expertise and written pedagogical communication is also a unique way of researching and understanding the work of MTEs.

METHODOLOGY

We used a grounded theory methodology (Charmaz, 2014) in this exploratory study to derive theoretical insights from qualitative data, without using pre-established theoretical models. This approach is consistent with Chapman's (2021) position that research on MTEs should extend beyond adapting pre-established models based on models of mathematics teachers' knowledge.

Data sources consisted of pedagogical mathematics booklets created by ten Brazilian MTEs, with experience ranging from seven to 30 years. Most of them, active in various national universities, hold PhDs in Mathematics Education and contribute to research in the field. The others, with PhDs in Mathematics, maintain a strong relationship with Mathematics Education. These experienced MTEs (referred to as MTEs) created 20 booklets in their role as advisors for a preservice mathematics teacher program at the newly established University of Federal District, Brazil, launched in the second semester of 2023. The booklets were intended for use in preparing PTs through direct mathematics-related teacher education courses or strategies and to support the new/Inexperienced MTEs (IMTEs) that would be hired by the university for the new education program. Thus, we viewed these booklets as encapsulating the MTEs' expertise, which this study sought to understand.

The booklets, situated in a mathematics-content education context, covered mathematics topics such as Numbers, Algebra, Geometry, Statistics, Probability, and Measurement. Each booklet consisted of separate information directed to the PTs and the IMTEs. For PTs, the information focused on tasks to develop their learning. For the IMTEs, the information expanded on the PTs' version to include guidance for them to use it in the teacher education program. These booklets demonstrate how the experienced MTEs who authored them articulated their pedagogical communication to PTs and IMTEs at the university. Thus, as qualitative data for this study, they provided insightful windows into the expertise of the MTEs.

Analysis of the data involved an emergent thematic approach through coding and axial categorisation (Charmaz, 2014). The focus was on identifying features of expertise among MTEs. Our analysis of the booklets resulted in identifying three principal categories associated with three different pedagogical contexts: mathematics, teaching mathematics and academic research. Each was characterized by its distinct purpose regarding the pedagogical content communicated in the booklets. Connections made among the categories were also identified within the booklets. These links emerged in the information directed at the IMTEs as a "bridge-building" feature of the expertise needed to engage the PTs in the mathematical activities/tasks. The three categories and the links formed four themes (three pedagogical contexts and bridge-building) that represent key features of the expertise for this group of MTEs, collectively, important to meaningfully prepare PTs in a mathematics-content education context.

FINDINGS

The findings are presented in terms of the four themes that represent key features of the expertise of the experienced MTEs related to preparing PTs through direct mathematics-related teacher education courses. These themes are framed in three pedagogical contexts and a process (bridge-building) that connects them.

MTEs' expertise as pedagogical context of mathematics. The pedagogical context of mathematics refers to tasks chosen by MTEs to broaden or deepen PTs' mathematical understanding. This context emphasises the selection or creation of tasks and their application in teacher education. Figure 1 exemplifies this pedagogical context.

Enter the geogebra link https://www.geogebra.org/m/d3sretyt
Regarding the graph of the linear function

$$f: R \rightarrow R; f(x) = ax + b; a, b \in R.$$

Vary *a* and *b*, in groups, analyse and answer:

(...)

d) What characteristic do you see in the graph when a > 0? And when a < 0?

e) If $a_1, a_2, \dots a_n$ positive real numbers are the angular coefficients of linear functions such that $a_1 < a_2 < \dots < a_n$, what is the relationship between the slopes of their graphs? What if the numbers are negative?

f) How would you justify the answers given in the previous items with more "formal" arguments?

Teacher Educator,

(...)

d) a > 0, the function is increasing, for a < 0, the function is decreasing. Teacher Educator, here you may need to recall (or ask students to look up) the definitions of increasing and decreasing functions.

e) It may be that the answer to this item is based solely on visualization. Encourage students to relate the slopes to the angles that the lines make with the Ox axis, and also to the fact that the tangent is increasing in its domain, i.e. in each subinterval of its domain.

f) Encourage students to increasingly formalize the arguments that justify their answers. In the following mathematical discussion, we present some concepts and a demonstration, which we suggest you discuss with the whole class.

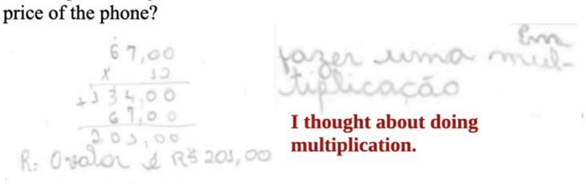
Figure 1: Excerpts translated to English from an Algebra booklet.

In Figure 1, the MTE presents a task requiring the PTs to use Geogebra for exploring the interplay between the parameters of a linear function's mathematical law and its graphical representation. Figure 1 also depicts a subsequent dialogue box for IMTEs, where the MTEs discuss potential responses, anticipate possible occurrences, and discuss interventions. This exemplifies the MTEs' expertise as pedagogical context of mathematics, which extends beyond task selection or design to include foreseeing the PTs' responses and planning pedagogical interventions. In general, the MTEs' written pedagogical communication indicated that their pedagogical-based mathematics content expertise includes a combination of knowledge of appropriate tasks to explore a mathematics concept, PTs' thinking in relation to the task, and intervention strategies to support the PTs' thinking about and learning of the concept.

MTEs' expertise as pedagogical context of teaching mathematics. The pedagogical context of teaching mathematics refers to the MTEs' selection of tasks to enhance PTs' experiential knowledge, focusing on aspects of teaching mathematics practice. This context involves engaging PTs in analysing curricular materials, student solutions, teacher narratives, and classroom observations, always contextualised within school practices. Figure 2 exemplifies this pedagogical context.

How do you think students reasoned in this resolution?

In a store window there is an advertisement: "SALE - Samsung Galaxy SII cell phone for just 12 installments of R$67.00". What is the total price of the phone?



The price is R$ 201.00

I thought about doing multiplication.

It would be very interesting to apply the problem in some sixth grade classes. This task could be carried out in partnership with primary school teachers. Another possibility would be to apply the problem to nephews, cousins, siblings, neighbors, children, collect the documents and analyze them collectively in the classroom, under the guidance of the teacher in charge.

In the discussions on analyzing the documents, it is important to point out that greater or lesser ease or difficulty in relation to algorithms is directly linked to understanding the operation, but also involves mastery of number structures and their articulation with the socio-cultural contexts in which they are applied and validated.

Figure 2: Excerpts translated to English from a Numbers booklet.

In Figure 2, the MTE introduces two narratives detailing fifth graders' approaches to a mathematical problem, prompting PTs to analyse the students' reasoning. The Figure 2 also shows how the MTE extends this initial task by suggesting tasks like applying the problem with students or family members and linking algorithmic challenges to operational understanding. This activity exemplifies a pedagogical context where school-based mathematical solutions are central. In general, the MTEs' pedagogical communication indicated that their pedagogical-based mathematics-teaching expertise includes designing tasks that concretely illustrate and allow PTs to engage in mathematics teaching from the school perspective, using real or hypothetical examples.

MTEs' expertise as pedagogical context of academic research. The pedagogical context of academic research refers to MTEs' communication of research findings directly with the PTs. This use of research goes beyond informing MTEs' teaching practices but involves explicitly integrating research outcomes into educational tasks with PTs, as depicted in the two examples in Figures 3.

The difficulty students have in understanding the different meanings associated with the use of the equal sign is well documented in the literature. For example, in a study of American students, Knuth, Stephens, McNeil and Alibali (2006) showed that students did not have a more sophisticated understanding of the equals sign and this is associated with their performance in solving equations. Some studies even point to the understanding of the equals sign in terms of equivalence as a predictor of students' later skills in algebra (MATTEWS; FUCHS, 2018). Even among teachers, there are difficulties in understanding the meanings of the equal sign, as in the research carried out by Trivillin and Ribeiro (2015).

Read the following texts carefully:

MONTEIRO, C. e PINTO, H. A Aprendizagem dos números racionais. Quadrante, v. 14, no.1, p. 89–107, 2005.

GUERREIRO, H. G. e SERRAZINA, M. DE L. A Aprendizagem dos Números Racionais com Compreensão Envolvendo um Processo de Modelação Emergente. Bolema, v. 31, no. 57, p. 181–201, 2017.

PINTO, H. A. e RIBEIRO, C., M. (2013). Diferentes significados das frações – conhecimento mobilizado por futuros professores dos primeiros anos. Anais... In R. Cadima, H. Pinto, H. Menino, I. S. Simões (Org.) Proceedings of the International Conference of Research, Practices and Contexts in Education, (pp.209-217). Leiria: ESECS.

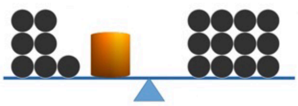
a) Produce a reflective synthesis of each of the texts.
b) Produce a text in which you bring together the main information about learning rational numbers, considering the role of the different interpretations (or meanings) of fractions in this process.
c) Prepare a presentation of your work for your colleagues.

Figure 3: Excerpts translated to English from booklets on Algebra and Numbers.

On the left of Figure 3, research presentation is integrated, as an argument, into a text written to be used in the PTs' learning. On the right of Figure 3, it is an activity for PTs to engage with three academic articles, synthesising and discussing their insights with peers. This approach aims to encourage PTs to distil findings and recommendations to their professional practice. In general, the MTEs' written pedagogical communication indicated that their pedagogical-based academic research expertise includes selecting pertinent mathematics education research for inclusion in teacher education, which can be done in various ways, ranging from citing research studies to engaging PTs directly with academic research.

MTEs' expertise as bridge-building. The bridge-building metaphor refers to MTEs explicitly making connections among the preceding mathematics, teaching mathematics, and academic research pedagogical contexts. For instance, in Figure 4, the MTE's pedagogical communication about linear equations involves introducing a two-pan balance model, which aligns with the mathematics pedagogical context. Although the situation is found in teaching practice, it was presented without this reference on this extract. However, a subsequent reference to a scholarly article critiquing the balance model's limitations matches with the academic research context. The MTE then points out how these limitations might be addressed in classroom, which now matches with the teaching mathematics context.

The scale in Figure 8 below represents the following question: "What number must be added to 7 to get 12?". Each ball on the scale represents 1 unit and the box represents the unknown value.



OTTEN, M.; HEUVEL-PANHULZEN, M. van den; VELDHUIS, M. The balance model for teaching linear equations: a systematic literature review. **International Journal of STEM Education**, v. 6, n. 1, p. 1-21, 2019.

In this study, the authors carried out a systematic review on the use of two-pan balance model in learning to solve linear equations. The advantage cited is that it provides a visual representation which helps to bring students closer to the solving algorithm. However, the same study points out the limitations of using the balance model to learn equations involving negative numbers.

The limitations presented above do not invalidate the use of scales to introduce solving 1st degree equations with one unknown, but they do alert teachers to the need to gradually move students from operations with scales to the symbolic register. For example, you could ask: "How could we represent this if we didn't have a two-pan scale? You can encourage students to make two parallel registers, one with the scales and the other symbolic, as we will illustrate below.

Figure 4: Excerpts translated to English from a booklet on Algebra.

Throughout the *corpus* occur other instances where the pedagogical contexts are interconnected in many of the booklets, mostly how topics are sequenced. Another example comes from one booklet which states that the IMTE should apply a mathematical task with PTs to enhance their mathematical understanding. This is followed by presenting school students' solutions for the same task, prompting teachers to discuss the students' reasoning and compare it to their own. This sequence, linking both the mathematics and the teaching mathematics contexts, illustrates a bridge between the two, prompting for relationships between teachers' mathematical solutions and student solutions. In general, the MTEs' written pedagogical communication indicated that their expertise includes a bridge-building process that involves inter-linking of ideas from the three different pedagogical contexts (between any two or among all three) to help PTs to understand the ideas in a connected way from different perspectives or contexts.

DISCUSSION AND CONCLUSIONS

This study contributes to the current call by researchers (Beswick & Chapman, 2015; Chapman, 2021; Helliwell & Chorney, 2022) to broaden the way we explore and understand MTEs' professional knowledge and practice. It demonstrates how a broader conception of MTEs' expertise that encompasses a fusion of consolidated experience, knowledge, and modes of participation, and MTEs' pedagogical communication to support PTs' learning could lead to understanding new features of MTEs' expertise. The findings suggest four of these features associated with three pedagogical contexts (mathematics, teaching mathematics, academic research) and a bridge-building skill.

The three contexts are based on how the MTEs structured their pedagogical communication, each serving a specific educational purpose. In the context of mathematics, the MTEs' expertise involved selecting, designing, and using mathematical tasks to enhance PTs' mathematical understanding, without direct reference to teaching practices. In the context of teaching mathematics, the MTEs' expertise involved choosing activities that directly address school teaching practices (real or hypothetical situations). In the context of academic research, the MTEs' expertise involved dissemination of findings from research for the purpose of informing teaching practice, ranging from informative texts to structured activities with research reports.

The MTEs also demonstrated an important bridge-building skill of making meaningful connections within and across the three contexts, based on their pedagogical communication. This bridge-building feature of their expertise indicated the ways in which they bridged the contexts to provide a meaningful basis for the PTs to engage with and develop deep understanding of them.

In conclusion, the study suggests that MTEs' expertise in preparing PTs through direct mathematics-related teacher education courses/strategies includes features associated with three pedagogical contexts. This study also suggests that MTEs' expertise includes a bridge-building feature used to integrate and navigate among multiple pedagogical contexts. This bridge-building approach represents the dynamic interplay between the different pedagogical contexts and the complexity of their professional knowledge. Thus, it provides a basis to address the complexity of MTEs' knowledge, which, as Chapman (2021) noted, requires more attention.

While this exploratory study has limitations in terms of the sample of MTEs and booklets used, it offers a basis to support future studies. For example, there needs to be further exploration of the four aspects of expertise with other MTEs and to include classroom observations of MTEs to understand these aspects from a lived perspective, particularly the bridge-building approach. Exploring this bridge-building approach is important to understand whether it exists in action, when and how it is used, and the nature and treatment of possible tensions that likely exist among the three different pedagogical contexts in trying to bridge them.

References

Bernstein, B. 2000. *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique* (Revised ed.). Oxford: Rowman & Littlefield Publishers.

Beswick, K., & Chapman, O. (2015). Mathematics teacher educators' knowledge for teaching. In S. J. Cho (Ed.), *Proc. 12th international congress on mathematical education: Intellectual and attitudinal challenges* (pp. 629–632). Berlin: Springer.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education*, 21(5), 417–427.

Beswick, K., & Chapman, O. (Eds.). (2020). *International handbook of mathematics teacher education, Volume 4: The mathematics teacher educator as a developing professional* (2nd Ed.). Leiden: Brill-Sense.

Chapman, O. (2021). Mathematics teacher educator knowledge for teaching teachers. In M. Goos & K. Beswick (Eds.), *The learning and development of mathematics teacher educators: International perspectives and challenges* (pp. 403–416). New York: Springer.

Chapman, O., Chitera, N., Climent, N., Dindyal, J., & Sztajn, P. (2022). Mathematics teacher education should be responsive to a rapidly changing world. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proc. 45th Conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 69-88). PME.

Coura, F. C. F., & Passos, C. L. B. (2021). Knowledge of the mathematics teacher educator who being a researcher of teaching. *Zetetiké*, 29, 1-18 – e021007.

Charmaz, K. 2014. *Constructing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

Goos, M., & Beswich, K. (Eds.). (2021). *The learning and development of mathematics teacher educators: International perspectives and challenges*. New York: Springer.

Helliwell, T., & Chorney, S. (2022). (Re)conceptualising the expertise of the mathematics teacher educator. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proc. 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 207-208). PME.

Martignone, F., Ferretti, F., & Rodríguez-Muniz, L. (2022). What aspects can characterize the specialised knowledge of a mathematics teacher educator? *Educación Matemática*, 34(3), 301-238.

Masingila, J., Olanoff, D., & Kimani, P. M. (2018). Mathematical knowledge for teaching teachers: Knowledge used and developed by mathematics teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education, 21*, 429–450.

Superfine, A. C., Prasad, P. V., Welder, R. M., Olanoff, D., & Eubanks-Turner, C. (2020). Exploring mathematical knowledge for teaching teachers: Supporting prospective teachers' relearning of mathematics. *The Mathematics Enthusiast*, 17(2/3), 367-402.

TEACHER NOTICING OF PRE-SERVICE AND IN-SERVICE SECONDARY MATHEMATICS TEACHERS – INSIGHTS INTO STRUCTURE, DEVELOPMENT, AND INFLUENCING FACTORS

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Teacher noticing is a crucial component of teachers' professional competence and has become a focus of educational research. However, evidence based on large-scale quantitative studies of the construct's structure, development, and influencing factors is scarce. Thus, in this paper, we briefly present the results of three recent studies that address these research gaps, summarize and discuss the findings, and formulate implications for future research. In the studies, we assessed noticing skills of pre-service and in-service teachers cross-sectionally and in a pre-post design, respectively, using an established standardised video-based instrument. Results shed light on the facet structure of teacher noticing, its development with increasing teaching experience, and the impact of learning opportunities in initial teacher education.

INTRODUCTION AND THEORETICAL BACKGROUND

To facilitate effective learning, it is essential for teachers to create rich learning environments characterized by high-quality instruction, effective classroom management, cognitive activation, and individualized support (Schlesinger & Jentsch, 2016). To achieve this, teachers must manage complex teaching situations characterized by sensory overload. They need to selectively perceive, comprehend, and respond to relevant instructional situations, which requires situation-specific skills commonly referred to as teacher noticing (Dindyal et al., 2021; Sherin et al., 2011). Over the past two decades, research on teacher noticing skills has steadily increased, particularly in mathematics education, as current literature reviews have shown (König et al., 2022). For example, researchers have investigated the characteristics of the construct (e.g. Mason, 2002; Sherin et al., 2011) and how to foster teacher noticing (e.g., Sherin, 2007). However, most studies are qualitative in nature, investigate only pre-service teachers, or treat small sample sizes (Amador et al., 2021; König et al., 2022). Moreover, standardised measurement of teacher noticing remains a challenge for the field, with few high-quality instruments available (Weyers et al., 2023). Thus, insight into the structure, development, and determinants of teacher noticing based on large-scale studies using standardised measures and quantitative analyses is lacking.

Conceptualisation of teacher noticing

Teacher noticing, sometimes referred to as professional vision, can be understood as a professionalized way of noticing that is characteristic of teachers in the classroom (Dindyal et al., 2021; Mason, 2002). Research on noticing is based on different

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 48–55). PME.

theoretical perspectives, including a cognitive-psychological, socio-cultural, expert-related, and discipline-specific perspective; the cognitive-psychological approach, which distinguishes cognitive processes that teachers engage in while noticing classroom events, is currently the most widely adopted perspective (König et al., 2022). Some recent studies also refer to the emergent embodied-ecological approach (Scheiner, 2021). Although some studies investigate noticing at a holistic level (Mason, 2002), most research has differentiated the construct into several facets that describe processes of observing, interpreting, and responding to classroom events important for teaching (Dindyal et al., 2021; Sherin et al., 2011).

In the Teacher Education and Development Study (TEDS) research program, we define teacher noticing as consisting of three facets: perceiving important classroom events, interpreting these events, and making decisions how to act (Kaiser et al., 2015). Thus, we include a decision-oriented facet in our framework to emphasise that noticing is strongly oriented toward classroom performance. We further understand teacher noticing as a situation-specific part of teacher competence. We conceptualise noticing as encompassing a general pedagogical and a mathematics pedagogical perspective, and include in our understanding a wide range of events relevant to quality mathematics education, such as students' thinking, classroom management or cognitive activation (Kaiser et al., 2015; Schlesinger & Jentsch, 2016).

Structure, development, and influencing factor of teacher noticing

Although many studies have distinguished facets of teacher noticing in their analyses, few have investigated the fit of their hypothesized construct structure. Seidel and Stürmer (2014) focused reasoning as an interpretation-related facet and demonstrated its distinguishability into description, explanation, and prediction. Yang et al. (2018) reported a good fit for a two-dimensional model that distinguished the general pedagogical and the mathematics pedagogical perspectives of teacher noticing. However, the empirical separability of the noticing facets perception, interpretation, and decision-making remains a research gap. Concerning the development of teacher noticing, longitudinal investigations are scarce. One exception is the study by Jong et al. (2021), who reported significant growth in perception and interpretation skills, but not in decision-making, over the course of a video-based university course for primary school teachers. In addition, a few studies have conducted cross-sectional comparisons of teachers with different levels of expertise or years of experience, which provides some initial evidence for the development of teacher noticing. For example, Jacobs et al. (2010) reported significant increases in perception and interpretation skills with increasing teaching experience and in the context of professional development, and for decision-making only from professional development. Overall, teaching experiences appear to facilitate the development of teacher noticing, although evidence on the extent and quality is lacking, and the current evidence is somewhat inconclusive (König et al., 2022). In addition, specific opportunities to learn (OTL) that may promote teacher noticing, particularly in the context of teaching experiences, have virtually not been investigated.

RESEARCH AIM AND QUESTIONS

In the light of the current discourse and the identified research gaps, in this paper we aim to investigate the structure and development of teacher noticing and the factors influencing it using standardised testing and cross-sectional as well as longitudinal analyses. Specifically, we address the following research questions: (Q1) What are the facets of teacher noticing and how are they related? (Q2) How do teacher noticing skills develop with increasing teaching experience? (Q3) Which OTL influence the development of teacher noticing? These three research questions have been addressed in three studies, described in the next section, as part of the TEDS research program.

METHODOLOGICAL APPROACH

To investigate the noticing skills of pre-service and in-service secondary mathematics teachers, we applied an established video-based instrument developed in the TEDS-Follow-Up study called TEDS-FU Video (Kaiser et al., 2015). The instrument consisted of three scripted (i.e., staged) video-vignettes (2.25 to 3.5 minutes in length) depicting lessons from the 9th and 10th grade. For each video-vignette, participants were given context information about the displayed lesson, then were allowed to watch the video-vignette once, and were asked to answer rating-scale and open-response items regarding their perception, interpretation, and decision-making from a general pedagogical or mathematics pedagogical perspective. The test consisted of 77 items (perception: *n* = 24, interpretation: *n* = 41, decision-making: *n* = 11). Figure 1 shows an example item.

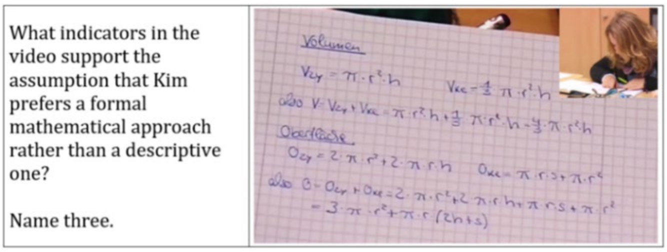

Figure 1: Open-response example item for the interpretation facet from a mathematics pedagogical perspective

Participants' responses were scored using extensive coding manuals, which had undergone an extensive expert review (Kaiser et al., 2015). Intercoder reliability for the open response items was satisfactory. We then estimated ability scores for all noticing facets using Rasch models. Reliability of the weighted likelihood estimates was good, except for somewhat low reliability for decision-making due to the small number of items and the complexity of the construct.

Study I

In the first study (Bastian et al., 2022), the sample consisted of 110 master's students (MS), 193 early career teachers (ECT), and 154 experienced teachers (ET) from Germany, who participated in studies of the TEDS research program (see descriptive statistics in Table 1). We investigated the empirical separability of the three noticing facets by contrasting the three-dimensional model of perception, interpretation, and decision-making with a one-factor model corresponding to a holistic understanding of noticing using Rasch models and structural equation modelling. In addition, we compared the three groups in their noticing skills using post hoc tests.

Variable	Master's students	Early career teachers	Experienced teachers
Sample size	110	193	154
Gender (% female)	66.4	58.3	54.5
School type (% academic track)	40.0	52.3	45.5
Teaching experience in years (M (SD))	0.0 (0.0)	4.6 (0.5)	19.6 (10.4)

Table 1: Descriptive sample statistics for the three groups from Study I and II

Study II

To further explore differences in specific cognitive demands, i.e., competence components related to the knowledge domains required to apply teacher noticing, between the three groups of teachers and to follow up on the findings from Study I, we further analysed the sample from Study I in Study II (Bastian et al., 2023). We developed an extensive category system to describe the cognitive demands of each test item, such as dealing with heterogeneity or competency-based teaching, and conducted multivariate analyses of variance to assess the relationship of the handling of these demands to the three experience groups.

Study III

In the third study (Bastian et al., under review), we complemented the two previous studies by measuring the teacher noticing skills of 175 MS (61.1 % female, M = 9.1 study semesters (SD = 2.7)) from six German universities before and after their teaching internship, thus enabling a longitudinal analysis of the development of their noticing skills. In addition, we assessed the OTL used by the MS during their internship, such as the extent of teaching practice, reflective activities, or mentor support, through a Likert item-based self-assessment after the internship. The data were then analysed using multiple regression and cross-lagged panel analyses.

RESULTS

The structure of teacher noticing

Based on the analyses of the MS, ECT, and ET in Study I, the three-dimensional model of perception, interpretation, and decision-making showed a better fit than a one-factor model ($\Delta\chi^2$ (Δdf) = 348.82 (2), $p < .001$), supporting the conceptual and empirical separation of the three facets (Bastian et al., 2022). The analyses revealed strong latent correlations between perception and interpretation (r = .81) and interpretation and decision-making (r = .82), but only moderate association between perception and decision-making (r = .46). This is consistent with the cross-lagged panel analysis from Study III (see Table 2). Interpretation at T1 had significant cross-lagged paths on perception and decision-making at T2 and predicted these skills, while the reverse cross-lagged paths on interpretation at T2 were not significant. Interpretation skills appeared to play a central role in the development of all three noticing facets and thus a prominent role in the structure of teacher noticing. The analyses provided evidence for a causal relationship, i.e., high interpretation skills prior to the teaching internships lead to higher perception and decision-making skills after the teaching internship.

Variable	Perception T2	Interpretation T2	Decision-making T2
	β	β	β
Perception T1	.36***	- .03	.02
Interpretation T1	.30***	.57***	.28**
Decision-making T1	-.15	.01	.36***
R^2	.33	.31	.34

Table 2: Cross-lagged panel model for the facets of teacher noticing

Note: ** ≙ p < .01, *** ≙ p < .001. T1/T2 ≙ measurement point 1/2. In the analysis, we controlled for high school diploma grade, semester, and dichotomized school type.

The development of teacher noticing skills

The comparison of MS, ECT and ET showed significant differences between pre-service and in-service teachers in favour of the latter for all three noticing facets, suggesting a significant development with teaching experience from MS to in-service teachers (see Table 3). However, this development does not seem to be linear, as ECT and ET only differed significantly in decision-making, but in favour of ECT, and the ECT performed nominally better in all three facets, indicating a possible decline in skills, but certainly saturation effects.

Variable	Master's students	Early-career teachers	Experienced teachers
Perception	45.6 (11.1)[a,b]	51.8 (9.2)[a]	50.1 (8.9)[b]
Interpretation	43.6 (12.0)[a,b]	52.6 (8.4)[a]	51.0 (8.5)[b]
Decision-making	44.0 (11.2)[a]	53.1 (8.4)[a]	50.3 (9.3)[a]

Table 3: Mean scores and standard deviation by experience group and facet
Note: For each facet, M = 50 and SD = 10. For each facet, values that differed significantly (at least p < .05) are indicated by the same letter.

This finding was further explored in Study II to better understand the areas, in which ECT and ET differ (Bastian et al., 2023). The results suggested significant differences with small effect sizes in three main teaching demands: decision-making (d = .34), the mathematics pedagogical perspective ($d = .28$), and recent mathematics pedagogical and general pedagogical topics ($.31 \leq d \leq .38$ [several demands investigated belonged to this category]), such as dealing with heterogeneity. Furthermore, Study III demonstrated a significant increase in a pre-post comparison over the course of the MS' teaching internship for all noticing facets with small effect sizes (perception: $d = .31$, interpretation $d = .39$, decision-making $d = .31$).

Influencing factors on teacher noticing

As reported above, interpretation skills and, to some extent, length of teaching experience appeared to be influencing factors in the development of all teacher noticing skills. In addition, we examined the influences of OTL on the noticing facets of MS in their teaching internships in Study III (Bastian et al., under review). These analyses revealed positive influences of linking theory to teaching situations on perception ($\beta = .19$*) and interpretation ($\beta = .25$***). Emotional mentor support during the internship ($\beta = .27$**) and time spent on lesson follow-up ($\beta = .19$**) significantly predicted decision-making. However, the amount of the MS' own teaching practice had no effect on the change of their noticing skills. This may suggest that deliberate, reflective practice and the conscious linking of theory and practice are necessary to facilitate the development of teacher noticing based on teaching experience gained in teaching internships.

DISCUSSION

In this paper, we presented the results of three recent studies on a quantitative standardised investigation of the structure, development and factors influencing teacher noticing. The analytical conceptualisation of teacher noticing with its three facets of perception, interpretation, and decision-making was empirically separable and proved to be superior to a single-factor model. In addition, decision-making was confirmed as an essential part of teacher noticing, not only through the model fit but also by its differentiating power on groups of teachers. This supports the adoption of analytical frameworks of teacher noticing that include a decision-oriented facet in future research. Furthermore, the findings of Study III indicated complex, reciprocal relationships between the three facets in their development, as opposed to a linear learning process of first learning to perceive, then to interpret, and then to make decisions. Interpretation appeared to be particularly important in the interactional structure, as it not only predicted its own development, but also influenced the development of perception and decision-making.

Teaching experience appeared to be a strong facilitator in the development of teacher noticing, as MS increased their skills over the course of their teaching internship, and ECT had significantly higher skills than MS. Therefore, a certain amount of teaching experience may be necessary to develop teacher noticing as a situation-specific skills. At some point, however, teaching experience alone may not be sufficient to progress further, as evidenced by the saturation or even decline in noticing when comparing ECT and ET. Deliberate practice may be necessary to further develop and achieve higher expertise in teacher noticing. This is supported by findings of Study III, which showed that time spent on lesson follow-up, the linking of the theories of university teacher education to specific situations, and the emotional mentor support had significant positive influences on facets of teacher noticing, while the mere amount of teaching had no effect by itself. Thus, deliberate practice in which teachers are aware of what they are doing, have time to reflect and link their practice to professional knowledge, and talk about their teaching with other teachers appears to be influential to develop noticing skills.

The studies have limitations that need to be considered and argue for cautious generalisation of the results. All three studies used convenience samples. Study I and II provided insights into the development of teacher noticing using cross-sectional data, so other influences, such as changes in teacher education curriculum, may have caused the observed differences. The analyses in Study III were conducted without a control group, so recall effects may have occurred.

Overall, the three studies provide new insights into the structure, development and determinants of teacher noticing based on quantitative analyses and standardised testing. However, more longitudinal studies and a closer look at the effects of OTL on its development are needed to better understand teacher noticing and to find improved means to foster its development.

References

Amador, J. M., Bragelman, J., & Superfine, A. C. (2021). Prospective teachers' noticing: A literature review of methodological approaches to support and analyze noticing. *Teaching and Teacher Education*, *99*, 1–16. https://doi.org/10.1016/j.tate.2020.103256

Bastian, A., Kaiser, G., Meyer, D., & König, J. (2023). The link between expertise, the cognitive demands of teacher noticing and, experience in teaching mathematics in secondary schools. *International Journal of Science and Mathematics Education*. https://doi.org/10.1007/s10763-023-10374-x

Bastian, A., Kaiser, G., Meyer, D., Schwarz, B., & König, J. (2022). Teacher noticing and its growth toward expertise: an expert–novice comparison with pre-service and in-service secondary mathematics teachers. *Educational Studies in Mathematics*, *110*(2), 205–232. https://doi.org/10.1007/s10649-021-10128-y

Bastian, A., König, J., Weyers, J., Siller, H.-S., & Kaiser, G. (under review). Effects of teaching internships on preservice teachers' noticing in secondary mathematics education. *Frontiers in Education*.

Dindyal, J., Schack, E. O., Choy, B. H., & Sherin, M. G. (2021). Exploring the terrains of mathematics teacher noticing. *ZDM – Mathematics Education*, *53*(1), 1–16. https://doi.org/10.1007/s11858-021-01249-y

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. www.jstor.org/stable/20720130

Jong, C., Schack, E. O., Fisher, M. H., Thomas, J., & Dueber, D. (2021). What role does professional noticing play? Examining connections with affect and mathematical knowledge for teaching among preservice teachers. *ZDM – Mathematics Education*, *53*(1), 151–164. https://doi.org/10.1007/s11858-020-01210-5

Kaiser, G., Busse, A., Hoth, J., König, J., & Blömeke, S. (2015). About the complexities of video-based assessments: Theoretical and methodological approaches to overcoming shortcomings of research on teachers' competence. *International Journal of Science and Mathematics Education*, *13*(2), 369–387. https://doi.org/10.1007/s10763-015-9616-7

König, J., Santagata, R., Scheiner, T., Adleff, A.-K., Yang, X., & Kaiser, G. (2022). Teacher noticing: A systematic literature review of conceptualizations, research designs, and findings on learning to notice. *Educational Research Review*, *36*. https://doi.org/10.1016/j.edurev.2022.100453

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.

Scheiner, T. (2021). Towards a more comprehensive model of teacher noticing. *ZDM – Mathematics Education*, *53*(1), 85–94. https://doi.org/10.1007/s11858-020-01202-5

Schlesinger, L., & Jentsch, A. (2016). Theoretical and methodological challenges in measuring instructional quality in mathematics education using classroom observations. *ZDM Mathematics Education*, *48*(1-2), 29–40.

Seidel, T., & Stürmer, K. (2014). Modeling and measuring the structure of professional vision in preservice teachers. *American Educational Research Journal*, *51*(4), 739–771. https://doi.org/10.3102/0002831214531321

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S. Derry (Eds.), *Video Research in the Learning Sciences* (pp. 383–395). Erlbaum.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing* (pp. 3–13). Routledge.

Weyers, J., König, J., Santagata, R., Scheiner, T., & Kaiser, G. (2023). Measuring teacher noticing: A scoping review of standardized instruments. *Teaching and Teacher Education*, *122*. https://doi.org/10.1016/j.tate.2022.103970

Yang, X., Kaiser, G., König, J., & Blömeke, S. (2018). Measuring Chinese teacher professional competence: adapting and validating a German framework in China. *Journal of Curriculum Studies*, *50*(5), 638–653. https://doi.org/10.1080/00220272.2018.1502810

TWO MORE OR TWICE AS MUCH? PROPORTIONAL REASONING STRATEGIES IN GRADES 5 TO 7

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Proportional reasoning determines the school performance of students not only in mathematics, but in other areas, and it plays an important role in everyday life as well. The early emergence of proportional reasoning is influenced by many factors, including the recognition of multiplicative relationships. In our research, we examined proportional reasoning by means of the interview method among 5th, 6th, and 7th-grade students. During the interview, in addition to solving proportional problems, the students solved open-ended problems that gave insight into their additive and multiplicative thinking. Our data and results can bring us closer to understanding the important requisites of proportional reasoning and the possible developmental step between additive and multiplicative reasoning strategies.

INTRODUCTION

Proportional reasoning is an important pillar of school mathematics. It is also an integral part of our everyday lives, whether it is cooking, shopping, pricing, or measuring distances. Proportional reasoning not only makes our daily lives easier, but it is also an essential foundation for many areas of mathematics. Proportional reasoning is a psychological construct with many structural connections to different fields of mathematics (Boyer and Levin, 2015). The level of development of proportional reasoning influences students' achievement in other school subjects (physics, chemistry, geography, etc.) as well. Although there are various views on the nature of the development of proportional reasoning, characteristics of the tasks used to measure the development play a crucial role in affecting students' solution strategies on proportional reasoning tasks (Fernandez et al., 2010). Our study aimed to map the presence of each of these conditions in grades 5 to 7. We interviewed the students and asked them to solve proportional task with different characteristics. The students solved the task while thinking aloud. The interview responses revealed students' thinking strategies in the sense of whether they are at the additive or multiplicative level of reasoning. In addition, the so-called "relative proportional reasoning", lodged right between additive and multiplicative thinking, was addressed (Gencturk et al., 2022). We investigated how the strategies used during the interview influenced the student's performance on each task.

THEORETICAL AND EMPIRICAL BACKGROUND

The age of onset of proportional reasoning was previously thought to be around the ages of 11 and 12 (Inhelder and Piaget, 1958). However, later research has shown that

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 56–63). PME.

proportional reasoning, or elements of it, are present from very early in life. Some forms of proportional reasoning are already present before the start of schooling and children develop the ability to interpret proportional situations early. For example, children aged 5 to 7 feeding differently sized fish gave more food to larger fish (Resnick and Singer, 1993). Different studies indicate different developmental rates, and there are different ideas about which types of tasks pupils can solve more effectively at different ages. Goswami's experiments with preschool children show that children as young as 3-4 years can use analogical reasoning with simple ratios. The experiment showed that young children can understand equivalent ratios even when quantities are not isomorphic (Singer et al., 2001)

The use of visual representations in task-solving situations allows the study of preschool and lower primary school age groups. They can be used to show elements of proportional reasoning without the use of concrete Arabic numbers or number words, or even special terms (e.g., ‘ratio’). Exercises based on visual representations can of course be applied to the upper primary age group and can be used to track proportional reasoning. Visual representations can represent both discrete and continuous quantities. Various studies have been conducted to investigate children's proportional reasoning with discrete and continuous quantities in preschool and primary school. Boyer and Levine (2015) found that preschool children are more adept at tasks with continuous quantities than with discrete quantities in a comparison problem (see also Boyer et al., 2008). Vanluydt et al.’s (2020) experiment with children aged 5-9 years shows that in missing-value problems, continuous quantities make it more difficult to solve tasks with discrete quantities compared to discrete quantities. Vanluydt et al.'s (2020) experiment with children aged 5-9 years shows that continuous quantities make it more difficult to solve problems with discrete quantities as opposed to missing-value problems.

In Jeong et al.’s (2007) experiment, 6-, 8-, and 10-year-old children were given a proportional reasoning task in the context of a game involving probability. These results suggest that the type of quantities used in the tasks (either discrete or continuous) strongly influences children’s ability to judge proportions. Proportional reasoning is influenced by further factors, such as one-to-many being easier for learners than many-to-many. In order to successfully solve proportional reasoning tasks, comprehension of general vocabulary and mathematics-related specific vocabulary is an important precondition. The presence of proportional reasoning requires knowledge of mathematical terminology, including specific vocabulary for proportional reasoning, such as double, half, etc. (Vanluydt et al., 2021).

An important pillar of children's development is the emergence of additive and then later multiplicative thinking. With the emergence of multiplicative thinking, students reach the level of what we may call true proportional thinking. One of the main challenges is to move from additive reasoning to multiplicative reasoning for proportional problems (Van Dooren et al., 2010) Degrande et al. (2018, 2019) investigated the extent to which third- to sixth graders prefer additive or multiplicative relationships by presenting them with problem situations that are equally open to both types of reasoning. It is predicted that multiplicative preference (as measured by multiplicative responses to open problems) will promote early proportional reasoning, while additive preference (as measured by additive responses to open problems) will act as a barrier, because children who prefer additive preference may make more additive errors in proportional problems (Vanluydt et al., 2022).

The question is whether the strategies used by children aged 10-13 can be categorized as strictly either additive or multiplicative or whether there is a transition between the two. In an experiment with teachers by Copur-Gencturk et al. (2023), it was found that there is a transition between the additive and multiplicative path, a third category labeled as "relative reasoning". Those characterized by the use relative reasoning strategy used the additive relationship between quantities but also paid attention to the size of the quantities and the differences between quantities. The shift from additive reasoning to proportional reasoning begins when one notices that the difference between two variable quantities does not remain constant as the size of the covariate quantities changes. Proulx (2023) used a series of interviews to study the proportional reasoning of two students, an 11-year-old boy and a 13-year-old girl. The interviews revealed the presence of a relative reasoning stage as a transition between additive and multiplicative reasoning.

Research questions

(1) What strategies do children use in upper and lower secondary grades while solving proportional reasoning tasks?

(2) How does strategy use affect students' performance on each task?

(3) How do task characteristics influence students' strategy use?

METHOD

Participants

In this study, 60 students were interviewed for 5th, 6th, and 7th grade classes (20 students from each grade) at a school in the capital city of Hungary.

Measures

Students were asked to solve two introductory and three proportional problems. In the introductory tasks, we gave two values for discrete (Point task) and continuous quantities (Bar task) in pictorial form and asked students to give the third element. When the third element was given, both additive and multiplicative reasoning could be correctly applied, in the latter case using a ratio of 1:2. In their mathematical studies at the lower secondary grades, students mainly encounter open-ended tasks given by discrete quantities. In the three proportional tasks, the ratio 2:3 appeared in different ways. In the first case (Dog task), discrete quantities were used, and the ratio was represented by a drawing. Students had to observe puppies and see what proportion of them were collared and uncollared. The second task (Orange task) was about comparing the amount of orange juice in two different measuring cups, thus solving a discrete quantity task (Proportion task). In the third task (Candy task), the price of some packets of candy was asked, and the data were given only in text or numbers. The two tasks used the same numbers.

While solving the tasks, we focused on the students' thought processes using asking

The students' interview responses were coded according to two criteria. We assessed the ability to solve the tasks and also the strategies used while solving the tasks. First, for each task, students' performance was assessed on a dichotomous scale (0 or 1). Strategies were coded according to the literature (Copur-Gencturk et al., 2023) as (1) additive reasoning, (2) multiplicative reasoning, (3) relative reasoning used by the student while solving the task, and (4) the use of a reasoning strategy different from the previous ones. Coding was done by two people according to the criteria discussed beforehand. In the case of divergent codes, the final position was established after discussion.

RESULTS

First, we looked after tasks by grade and overall. Table 1 presents the mean (SD in parentheses) performance of students in different grades on the items.

Item	Grade 5	Garde 6	Grade 7	Total
Points	1.00 (.00)	1.00 (.00)	1.00 (.00)	1.00 (.00)
Bar	.25 (.44)	.45 (.51)	.60 (.50)	.43 (.50)
Dog	.50 (.51)	.55 (.51)	.70 (.47)	.58 (.50)
Orange	.15 (.37)	.20 (.41)	.55 (.51)	.30 (.46)
Candy	.80 (.41)	.90 (.31)	.90 (.31)	.87 (.34)

Table 1: Students’ mean performance (SD in parentheses) on the items according to grades.

The ANOVA test shows that in four tasks (Points, Bar, Dog, and Candy) there is no significant difference between the performance of the grades. However, in the Orange task, there is a significant difference between grades 5 and 7; $F(2,57) = 5.061$, $p = .009$, (Levene's test for homogeneity, $p = .002$, Dunnette T3 post-hoc comparison gives $p = .022$).

Strategies

We looked at the strategies students used in each task. The frequency of occurrences is shown in Table 2 for each task.

Item/Strategy	(1) Additive	(2) Multiplicative	(3) Relative	(4) Other
Points	52	8	0	0
Bar	22	19	6	13
Dog	1	27	23	9
Orange	44	16	0	0
Candy	7	53	0	0

Table 2: The appearance of strategy types per task

Additive and multiplicative strategies were the most frequently used. However, there is a remarkable appearance of relative reasoning. The emergence of this strategy was observed only in the closed tasks. We observed two types of relative reasoning. First, when learners realize that it is not enough to consider the difference in quantities, but that there is more to it than that. They go beyond additive reasoning but do not know how many times they should be looking. When the second type appears, the learner is already aware that it is a multiple but does not know the calculation procedure or cannot calculate the result using the multiplicative strategy. In this case, they use it as a kind of "escape route", trying to find an analogy in the solution.

One student gave the following answers after giving a good solution:

S: There are 15 dachshunds and 6 of them are collared. I think this is the right one.

I: Why do you think that?

S: Because that's how I think it would work out proportionally. Yes. So proportionately.

I: What do you mean, proportionally?

S: If you look at it, it's the most similar. Yes, because here it's 4 and then there's only 1 with a collar and it's not so similar proportionally, but here it's about the same proportionately.

I: And if you counted?

S: 15 divided by 6 is about 2, and then let’s say 3:2 is one. Then I don't know. By counting it seems to be closer, because 4:1 is further away. Anyway, I don't know. I think it looks right.

For the open tasks, we found that students did not use relative reasoning, their solutions were limited to the other options: additive reasoning and multiplicative reasoning. The proportional task (Csapó, 1997) was the most characteristic example of the separation of learners' reasoning in terms of additive and multiplicative reasoning. In this task, only these two types of thinking appeared. If one scale goes up by two, the other will also go up by two. Those who used additive reasoning were convinced of the correctness of their solution. Those who used multiplicative reasoning were more uncertain about their solution. When asked what would happen if half of the orange juice was poured into the jar, the students corrected their solution after the comparison and realized that the additive solution was not the right way to go.

The Candy task had the same numbers as the Orange task, yet the students performed better there. Those who recognized that they were doing an analogous task sometimes corrected their previous solution. It may have been easier in this task because the amount of sugar and the amount to be paid were different units of measurement, but in the orange juice task, the equal distance marked on two pots of different widths did not mean different measurements to the children.

Does the right solution depend on the choice of strategy?

Everyone solved the Point task correctly, so there was no difference in the strategies used. In the Band task, there is a significant difference in the correctness of the solutions between groups using different strategies, as confirmed by ANOVA, $F(3,56) = 3.350$, $p = .009$ (Levene's test for homogeneity, $p < .001$; Dunnette T3 post-hoc comparison gives $p = .025$). There is no significant difference between those using additive and multiplicative solutions. However, there is a significant difference in the ability to solve the task between those using additive vs relative and multiplicative vs relative strategies (based on the results of the Levene test for homogeneity, $p < .001$ Dunnette T3 post-hoc comparison gives $p < .001$ and $p = .011$).

In the Dog task, only one student used the additive reasoning strategy. However, there is a significant difference between the multiplicative and relative strategy users in their performance. For those using relative reasoning, the ratio of correct answers was 52%, and for those using multiplicative reasoning, it was 82%. An independent samples t-test was used, Levene's test is significant, $p < .001$, i.e. equal variances are not assumed, so the Welch-test is significant for the difference between the performance of the two groups, $p = .031$. Students using the relative reasoning strategy performed worse compared to students using the multiplicative reasoning strategy.

In the orange task, 94% of those using the multiplicative strategy solved the task correctly. In the candy tasks, additive and multiplicative reasoning are typical. 94% of the students using multiplicative reasoning gave the correct answer, while one student got the right solution by another method.

Consistency or preference in strategy use

The prevalence of strategies used in each task does not typically show clear preferences in this age group. A crosstabs analysis was conducted to compare the strategies used in the tasks, showing a significant contingency coefficient between strategy use in the two cases. For the Point task - Dog task pair, $p = .047$, and for the Dog task - Orange task, $p = .013$.

DISCUSSION

The main novelty of the current research is addressing and documenting students' proportional reasoning according to task variables. Our most important finding concerns the presence of relative reasoning that can only be observed in closed tasks, and not in open-ended tasks. Relative reasoning is a kind of transition between additive and multiplicative strategies (Proulx, 2023). The cross-tabulation analysis shows that those who used relative reasoning on closed tasks often switched to additive reasoning on open tasks as if they moved back to an earlier stage of the developmental ladder.

Among the task characteristics, the use of continuous versus discrete quantities proved to be also relevant. In the case of continuation of points and discrete quantities, the students solved the problem without error. However, in the bar task, for continuous quantities, when they were asked to continue bars in a way of their choice, the accuracy was .43 (.50). At this age, students seemed to have more difficulty on the continuous quantity task. The task was presented as a missing-value problem and supports the results obtained at lower ages, which for this type of task were found to be more difficult than discrete tasks (Vanluydt et al., 2020).

The most important educational implications are as follows. Through discussion of several closed tasks, the path towards multiplicative reasoning could be reinforced by acknowledging and taking advantage of relational reasoning as an intermediate step. Thus, the use of relative reasoning could be a powerful educational strategy in the teaching of ratios since relationality could reinforce the recognition of ratios. In the classroom environment, peer interactions could facilitate the fostering of proportional reasoning. Fielding-Wells et al. (2014) investigated the development of fourth-grade students' proportional reasoning in a classroom setting and found that proportional reasoning helped transform additive strategies into proportional (multiplicative) reasoning. Peer interactions improve students' solution processes and explain individuals' subsequent gains (Schwarz and Linchevski, 2007).

Additional information

This work was supported by the Research Programme for Public Education Development of the Hungarian Academy of Sciences.

REFERENCES

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: where young children go wrong. *Developmental Psychology*, *44*(5), 1478.

Boyer, T. W., & Levine, S. C. (2015). Prompting children to reason proportionally: Processing discrete units as continuous amounts. *Developmental Psychology*, *51*(5), 615.

Csapó, B. (1997). The development of inductive reasoning: Cross-sectional assessments in an educational context. *International Journal of Behavioral Development*, *20*(4), 609-626.

Copur-Gencturk, Y., Baek, C., & Doleck, T. (2023). A closer look at teachers' proportional reasoning. *International Journal of Science and Mathematics Education*, *21*(1), 113-129.

Degrande, T., Verschaffel, L., & Van Dooren, W. (2018). Beyond additive and multiplicative reasoning abilities: How preference enters the picture. *European Journal of Psychology of Education, 33*, 559–576.

Degrande, T., Verschaffel, L., & Van Dooren, W. (2019). To add or to multiply? An investigation of the role of preference in children's solutions of word problems. *Learning and Instruction, 61*, 60–71.

Dooren, W. V., Bock, D. D., & Verschaffel, L. (2010). From addition to multiplication… and back: The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction*, *28*(3), 360-381.

Fernández, C., Llinares, S., Modestou, M., & Gagatsis, A. (2010). Proportional reasoning: How task variables influence the development of students' strategies from primary to secondary school. *Acta Didactica Universitatis Comenianae Mathematics*, Issue 10. 1-18.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. Routledge.

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous versus discrete quantities. *Journal of Cognition and Development*, *8*(2), 237-256.

Fielding-Wells, J., Dole, S., & Makar, K. (2014). Inquiry pedagogy to promote emerging proportional reasoning in primary students. *Mathematics Education Research Journal*, *26*, 47-77.

Proulx, J. (2023). Relative Proportional Reasoning: Transition from Additive to Multiplicative Thinking Through Qualitative and Quantitative Enmeshments. *International Journal of Science and Mathematics Education*, 1-22.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In: Carpenter, T. P., Fennema, E. & Romberg, T. A. (Eds.), *Rational numbers: an integration of research* (pp. 107-130). Lawrence Erlbaum Associates.

Singer-Freeman, K. E., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates?: Proportional matching in an analogy task. *Cognitive Development*, *16*(3), 811-829.

Schwarz, B. B., & Linchevski, L. (2007). The role of task design and argumentation in cognitive development during peer interaction: The case of proportional reasoning. *Learning and Instruction*, *17*(5), 510-531.

Vanluydt, E., Degrande, T., Verschaffel, L., & Van Dooren, W. (2020). Early stages of proportional reasoning: A cross-sectional study with 5-to 9-year-olds. *European Journal of Psychology of Education*, *35*, 529-547.

Vanluydt, E., Supply, A. S., Verschaffel, L., & Van Dooren, W. (2021). The importance of specific mathematical language for early proportional reasoning. *Early Childhood Research Quarterly*, *55*, 193-200.

Vanluydt, E., Verschaffel, L., & Van Dooren, W. (2022). The early development of proportional reasoning: A longitudinal study of 5-to 8-year-olds. *Journal of Educational Psychology*, *114*(6), 1343.

EXPLORATIONS WITH AMBIGUITIES IN MATHEMATICAL PROBLEM-SOLVING

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This study explores the characteristics of narratives in a problem-solving discourse, where the reasoning processes of two students are analysed. It aims to examine the factors contributing to endorsing students' generated narratives during the process. The results indicate that explorative actions are characteristic of the narratives generated by the students. The primary factors contributing to the endorsement of narratives are identified as "Ambiguity of difference in sameness" and "Ambiguity of generalization". Awareness of these different ambiguities, their nature, and their role in the discourse is crucial for how mathematics teacher educators can support students' development in mathematical reasoning.

INTRODUCTION

It is important for teacher students to develop and enhance their problem-solving skills. Given that problem-solving constitutes a fundamental aspect of the high school mathematics curriculum, achieving proficiency in this subject becomes essential for educators. Teachers play a pivotal role in guiding their students, not only to understand mathematical concepts but also in explaining, coordinating, and demonstrating the proper use of mathematical terminology as defined by the discourse within the relevant mathematical community. Following Sfard (2008), this paper uses the term "discourse" to describe the specific form of communication that is used by mathematicians or used within mathematics classrooms. This form of communication is characterized by endorsed narratives expressed through mathematical definitions, theorems, or proofs considered accurate when accepted by the participants in the discourse.

The study presented in this paper is part of an ongoing project exploring students' mathematical reasoning in problem-solving, presenting a small portion of empirical data involving a pair of students working on a mathematical maximization problem in Calculus. Before this study, the reasoning of another student pair dealing with the same problem was analysed. Observation in the reasoning of the first student pair, revealed that ambiguities created by the students played a crucial role in how their narratives were accepted. Using Sfard's research (2008, 2021, 2023), this paper aims to characterizing the narratives of the second student pair and identify the factors contributing to the approval of both students' narratives. The question addressed in this paper is: *What are the characteristics of narratives in a problem-solving discourse involving derivative, and what factors contribute to the endorsement of these narratives?*

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 64–71). PME.

RELATED LITERATURE

Understanding Calculus and derivatives is a crucial aspect of mathematics learning for preservice teachers. Studies in Calculus constitute a significant part of Swedish high school mathematics and, consequently, a substantial component of teachers' future professional practice. However, previous research has indicated a limited understanding of derivatives among many students (e.g., Haghjoo et al., 2020). It is also common for students to prioritize symbols over graphical representation when studying derivatives (e.g., Biza, 2021). Despite existing research on students' reasoning in Calculus, there seems to be a need for additional investigations into discourses involving Calculus and problem-solving in university courses for prospective teachers in upper secondary school (e.g., Mukuka et al., 2023).

In this paper, discourse is used in line with Sfard's (2008) definition which also include the notion of endorsed narratives. In the endorsement process unacknowledged discursive gaps may arise where some of them are essential for learning while others are more inhibitory (Sfard, 2023). Sfard also uses the term ”communicational gaps” to describe a critical aspect that is important for teachers to be aware of in their communication with students in mathematical discourse (Sfard, 2021). One critical aspect of communication is ambiguities, as observed in the study by Berggren et al. (2023), which indicates that students exhibit ambiguity several times when working together on a Calculus problem. Investigating these ambiguities provides an opportunity to gain a deeper understanding of the students' reasoning process. According to Barwell (2003), ambiguity can allow students to investigate what and how it is possible to express with mathematical language and explore mathematics. Peterson et al. (2020) define "clarifiable ambiguity" as an ambiguity that essentially arises when someone utilises an unclear referent. When asked, the person responsible for creating this ambiguity can straightforwardly provide clarification (Peterson et al., 2020). In this paper, the term ”ambiguity” is used to describe an essential aspect of communicational gaps when two students discuss a mathematical problem they are trying to solve.

THEORETICAL FRAMEWORK

For the analysis of the data, consideration is given two characteristics of mathematical discourses: narratives and routines (Sfard, 2008). According to Sfard (2008), there are a total of four characteristics: word use (mathematical vocabulary), visual mediators (visible objects), narratives (any sequence of utterances that describe objects, relations, and process, such as definitions, theorems, and proofs), and routines (repetitive patterns characteristic of mathematical discourse). Mathematical routines develop through repetition and adaptation of previous practices (Sfard, 2008). They involve defined and recurring patterns in mathematical discourse, guiding tasks such as calculation and proof. According to Sfard (2008), there is three categories of mathematical routines: explorations, deeds, and rituals. A routine becomes an exploration if its implementation contributes to a mathematical theory, such as, for example, an axiom, definition or theorem. Deeds, however, focus on producing actions or modifying mathematical objects. Rituals are routines aimed at creating and sustaining a bond with other people while recognizing their authority within the discourse. Exploration routines can be further subdivided into three distinct subtypes: construction, substantiation, and recall (Sfard, 2008). Constructions result in new narratives that are endorsable. Substantiations are actions that help students decide whether to endorse or reject constructed narratives. Recalls are the processes one performs to be able to summon a previously endorsed narrative.

METHOD

This study is qualitative and analyses the communication between two male students working together on a mathematical problem-solving task (Figure 1, translated from Swedish to English). In the problem-solving situation, the students engage in reasoning.

At a point on the curve, $y = e^{-x}, x \geq 0$ a tangent to the curve is drawn.

The coordinate axes together with the tangent form a triangle.

What is the largest area such a triangle can have?

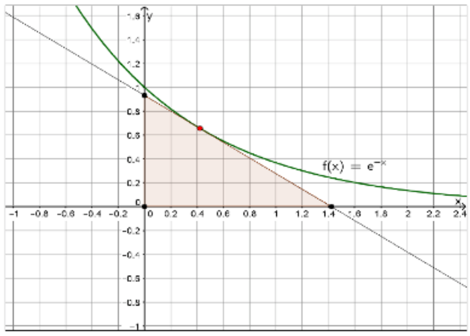

Figure 1: The task, as presented to the students (Berggren et al., 2023)

The student pair is the second of two pairs whose communication is analysed. The various communications between the two pairs of students could be seen as two different mathematical discourses. With other discourses come different routines and, thus, different ways of acting (Sfard, 2021). I have used the same research question for both pairs of students to obtain a more significant variation of characters in a narrative within a problem-solving discourse involving derivatives. The analysis of the different pairs of students is part of a dissertation investigating the mathematical reasoning of preservice teachers.

The data presented in this paper were collected during a mathematics course on Calculus for preservice teachers in their first year of their mathematics teacher education program. They work in pairs with a task chosen by the researcher among a set of tasks provided by the course teacher. The students worked without the supervision of the teacher and outside the classroom. The data consists of the students' written answers and a videorecording capturing the students' hands and actions with paper and pencil during the session.

I used the transcription of what the students were saying and writing and analysed every sentence and action in part. The beginning of the analyse consisted of identifying key aspects of the discourse: word usage, visual mediators, and routines, and their interrelationships. Episodes was then divided into constructed or endorsed narratives. The review of preliminary narratives was closely aligned with the study's theoretical framework and the research question.

RESULTS AND ANALYSIS

The students work on the problem in a logical sequence of sub-problems that they encounter. Parts of the students' work are presented in the order they approached them. The entire reasoning is divided into four distinct narratives: *The area of the triangle*, *The slope of the tangent line*, *The intersection of the tangent line and the y-axis* and *The equation of the tangent line*. Three of these narratives involve ambiguities, which will be presented below.

Just before the following discussion, the students have determined, with the aid of a computer program (GeoGebra), that the triangle has its maximum area when the x-coordinate of the point of tangency is equal to one. They have also chosen to denote this x-coordinate as a. The students are now attempting to utilize a specific case, when the point of tangency is $(1, e^{-1})$, to derive the equation of the tangent.

The area of the triangle

36	S1:	[…] but then we think that… based on this... and that is when a is equal to one… it seems like… or when?
37	S2:	You mean the height is equal to one, right?
38	S1:	What? No, but the point.
39	S2:	The point… yes. Okay.
40	S1:	Or the x-coordinate one.
41	S2:	But isn't it a bit strange to have it as a when we have the area as a? …so can we name it P for example?
42	S1:	It sounds better. Do you think that…
43	S2:	That's approximately one, isn't it? It's not exactly one, is it?
44	S1:	It says…
45	S2:	At one, okay? [Writes $P = 1$]…it is good.

In rows 37-40, the students attempt to ensure that they are referring to the same object. The language is clarified, and they agree that they are talking about the x-coordinate of the point of tangency. In row 41, student 2 realizes the need for a new notation for the point they are discussing. Through substantiation using GeoGebra, the students then attempt, in rows 43-44, to determine whether the maximum area for the triangle formed by the tangent and the coordinate axes occurs precisely when the x-coordinate of the point of tangency is one. When student 2 writes $P = 1$, the narrative is accepted. In this excerpt, an ambiguity emerges that the students themselves do not seem to be aware of yet. What does the documentation $P = 1$ imply for the students' ongoing reasoning? $P = 1$ is an unclear reference as it does not specify which coordinate the "1" is referring to. This can be interpreted as an ambiguity denoted as *ambiguity of* *different in sameness*. The term *different* means that distinct objects are referred to, and *sameness* pertains to the use of a single notation. In this case, it means that students are using a single notation for two different objects. The symbol P is employed for both the x- and y-coordinates.

The intersection of the tangent line and the y-axis

The students use the slope-intercept formula for a line, which is expressed as $y = kx + m$, were k is the slope of the line, and m is the y-intercept (following standard Swedish notation). They are now trying to find a value for m. Student 2 has previously written that $h = \frac{1}{e}$, referring to the height of the point of tangency.

67 S2: And it is the height when… when x is zero then it must be the height.

68 S1: Exactly, because it is then m.

69 S2: But when we have e to the power of x… to the power of zero, then it becomes one. [Point at the expression $h = \frac{1}{e}$ Which the student had written earlier.] …so, then the m-value is to the power of minus one, right...? We have e to the power of zero and it is equal to one. [Writes $e^0 = 1$.]

70 S1: mm.

71 S2: So, what we have is the height. It is then equal to m minus one. [Writes $h = m - 1$.]

72 S1: Because now we put... We'll see here...

73 S2: No.

74 S1: For now… exactly, yes, that one won't be right [Points at $h = \frac{1}{e}$ which student 2 has written.]

75 S2: No. [Adjusts the expression to $h \neq \frac{1}{e}$]

The utterance in row 69 indicates that student 2 imagines the point of tangency moving along the tangent instead of along the curve $y = e^{-x}$. Student 2 looks at an expression ($h = \frac{1}{e}$) that is not related to the graphical context they are discussing. The mistake that occurs can be traced back to naming the y-coordinate of the point of tangency as h, which is the same notation used for the height of the triangle. In row 74, student 1 shifts focus from the graph to the algebraic expression on paper and realizes that these two do not align. The *ambiguity of different in sameness* appears when the same notation is used for both the height of the triangle and the height of the point of tangency. An additional challenge involves the students struggling with generalizing expressions. This is observed as students' struggle with finding a value for m instead of seeking a general expression.

The Equation of the Tangent Line

The students have derived an equation for the tangent in a specific case and determined that $k = -\frac{1}{e}$ and $m = \frac{2}{e}$. When the students then attempt to express the value of m for a general tangent to the curve, they start with the expression $m = \frac{1}{e} - \frac{-1}{e} \cdot x$ which yielded $m = \frac{2}{e}$ when $x = 1$.

125 S2: It will be x everywhere, right?
[Adjusts the expression for m to $m = \frac{1}{e^x} - \frac{-1}{e^x} \cdot x$]

Here, the *ambiguity of different in sameness* becomes evident. The students substitute x instead of the specific x-coordinate, 1, used in the particular case. This results in a notation representing two different objects in the expression. At this moment, the students do not seem to be aware of the ambiguity.

Subsequently, the students work on simplifying and rephrasing the expression for m in various ways, attempting once again to substitute $x = 1$ into the expression for m. After an eight-minute discussion, student 1 recognizes the ambiguity.

178 S1: […] It feels like we need to have two x-values or something.

179 S2: We shouldn't need that.

Student 1 is interrupted by the utterance, "We shouldn't need that," and both students agree to start over with the equation of the tangent. They now opt to define the point of tangency P as $P = (x, y)$.

181 S2: […] Now we set the point P with an x and a y value.

This indicates that the students perceive ambiguity in the previous definition of P. They proceed to search for a general expression for *m* but now begin to articulate their intentions more clearly to each other. Following another seven minutes of discussion, they initiate a discussion about the point of tangency and its x-coordinate in relation to the tangent's intersection with the x-axis.

226 S1: So now we then want to find that x-coordinate based on that x-coordinate. [Points first to the point of intersection of the tangent with the x-axis and then to the point of tangency.]

The student seems to understand that there are two different objects, yet continues to use the same notation for both. After a brief discussion about the tangent's intersections with the coordinate axes, student 1 realizes what needs to be done.

234 S1: […] but then we have to... We have to differentiate... the mistake we made... We kind of got that when we came back from the beginning because we used x in too many places I think. Because we use x for that x and x for that x... I think...

Student 1 engages in a reasoning process with themselves here and realizes the mistake of using the same notation for two different objects (the x-coordinate of the point of tangency and the x-coordinate of the tangent's intersection with the x-axis). The students go beyond the ambiguity by introducing two different variables for the two distinct objects. In this and previous narrative an additional ambiguity appears which is denoted as *ambiguity of generalization.*This ambiguity characterizes of the students' not using a mathematical notations that allows them to transition from the specific to the general case. Repeatedly the students encounter challenges as they tend to substitute specific values into the general expressions, preventing them to advance in the problem-solving process.

DISCUSSION AND CONCLUSION

The students use rituals to confirm their involvement in the discussion as they proceed. The discussion is solution-focused and they express themselves minimally, both linguistically and mathematically. Nevertheless, the ambiguities in their communication are observable with two prominent ambiguities in their narratives: the *Ambiguity of different in sameness* and the *Ambiguity of generalization*.

The discourse is explorative, with ambiguities arising both as a part of and a consequence of the students' exploration routines. These ambiguities are challenges that must be addressed before the students can make progress in their further reasoning. It is crucial for the teacher to identify what these ambiguities are and understand the reasons for their occurrence. Ambiguities play an important role in the students' argumentation training and mathematical theory use. Often, they are necessary for students to continue in an explorative phase of the discussion. The explorative phase refers to the part of a narrative that consists of exploration routines.

The result indicates that ambiguities, as aspects of communication gaps, have significance in how narratives are endorsed. In the empirical context, narratives with ambiguities can be interpreted as challenges, necessitating students to employ explorative routines such as constructions, substantiations, and recalls to reach a solution. The ambiguities that arise often depend on how mathematical symbolic language is utilized (Barwell, 2003). The task under discussion requires students to use mathematical notations. Despite being perceived as solution-focused, their reasoning becomes more explorative than ritualized and the students are compelled to delve deeper into their descriptions of mathematical objects.

There are several "clarifiable" ambiguities (Peterson et al., 2020) throughout the entire reasoning process. However, the ambiguities identified in this study differ, as they do not appear as easily "clarifiable" to the students, making resolution less straightforward. Nevertheless, addressing these ambiguities is crucial for the students to make progress. Additionally, these ambiguities do not always involve situations where it is unclear what the students are referring to. Instead, visual mediators like pictures and graphs are used to clarify the mathematical objects they are discussing.

Recognizing the role of ambiguities can contribute to better understanding of students' process of narrative endorsement and what new narratives that they need to create. Awareness of the different ambiguities, their nature, and their role in the discourse is essential for how mathematics teacher educators can support students' development in mathematical reasoning. The teacher's knowledge of the various types of ambiguities can also make students aware of how a problem-solving process can be productive for learning mathematics.

References

Barwell, R. (2003). Ambiguity in mathematics classroom discourse. In J. Williams (Ed.), *Proceedings of the British society for research into learning mathematics* (Vol. 23, pp. 1–6). University of Manchester.

Berggren, E., Olteanu, C., Perez, M., & Sollervall, H. (2023). Praxeologies-in-action during struggle with problem-solving. In Drijvers, P., Csapodi, C., Palmér, H., Gosztonyi, K., & Kónya, E. (Eds.). (2023). *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 2267–2274). Alfréd Rényi Institute of Mathematics and ERME.

Biza, I. (2021). The discursive footprint of learning across mathematical domains: The case of the tangent line. *The journal of mathematical behavior,* 62. https://doi.org/10.1016/j.jmathb.2021.100870

Haghjoo, S., Reyhani, E., & Kolahdouz, F. (2020). Evaluating the understanding of the university students (Basic Sciences and Engineering) about the numerical representation of the average rate of change. *International journal of educational and pedagogical Sciences, 14*(2), 111–121. https://doi.org/10.6084/m9.figshare.12488879

Mukuka, A., Balimuttajjo, S., & Mutarutinya, V. (2023). Teacher efforts towards the development of students' mathematical reasoning skills. *Heliyon, 9*(4), e14789. https://doi.org/10.1016/j.heliyon.2023.e14789

Peterson, B. E., Leatham, K. R., Merrill, L. M., Van Zoest, L. R., & Stockero, S. L. (2020). Clarifiable ambiguity in classroom mathematics discourse. *Investigations in Mathematics Learning, 12*(1), 28-37.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge university press.

Sfard, A. (2021). The devil in details: Mathematics teaching and learning as managing inter-discursive gaps. *In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo & S. L Chin (Eds.), Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia), pp. 1-18. Singapore: MERGA.*

Sfard, A. (2023). The devil's finest trick: routines that make teachers matter against their better judgement. *Journal of Curriculum Studies, 55*(1), 21-36. https://doi.org/10.1080/00220272.2023.2172359

PARENT PERCETIONS OF THEIR MATH PARENTING ROLES IN THE HOME MATH ENVIRONMENT

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Type A growing body of research has demonstrated the critical importance of the early home mathematics learning environments of children prior to the onset of formal schooling in kindergarten. However, very few studies have looked specifically at factors that influence the motivations and decision-making of parents with respect to their math parenting. This study used the RESET framework to examine the perceptions of parents (n = 847) of 4- to 5-year-old children who were not yet in kindergarten to better understand how they perceived their role and actions in the HME. Better understanding of parents as agents in the HME can inform stakeholders desiring to improve the success of home intervention and support programs.

INTRODUCTION & BACKGROUND

A large body of research supports the impact of early math success on later learning (Claessens & Engel, 2013; Duncan et al., 2007; Nguyen et al., 2016; Ritchie & Bates, 2013; Schweinhart et al., 2005). Success in school math is associated with higher rates of secondary graduation, college entry, college graduation, lifetime earnings, and more (Schweinhart et al., 2005). Unfortunately, the math learning of students in the USA is in crisis (deBrey et al., 2019; OECD, 2023). International assessments of math performance show USA students seriously lagging behind other countries, many of whom have just a fraction of the GDP of the United States (OECD, 2023). The National Assessment of Educational Progress (NAEP) – a standardized assessment that has been administered to 4th, 8th, and 12th grade students since the 1960s—reveals that more than 60% of students not proficient in grade level standards (deBrey et al., 2019). And while the past two decades have shown a gradual upward trend in mathematics achievement, fully all gains were erased due to the Covid-19 pandemic, returning the general population of learners back to 2003 math achievement levels (Bryant et al., 2023). The United States, long known as a global leader in the fields of mathematics, science, and STEM, is not likely to remain so if the rising generations are not proficient in mathematics.

Many children enter formal schooling without the background knowledge needed to successfully learn (Betts, 2021; Betts & Son, 2022). These differences in readiness-to-learn result in some children being able to make the most of the learning that takes place in school, while others struggle to learn and master key kindergarten math competencies. Such lost learning opportunities, in turn, contribute to persistent and widening gaps in students' math learning as they move from grade to grade (Duncan et al., 2007). And while some children are enrolled in formal preprimary schooling

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 72–79). PME.

options (e.g., preschool, pre-kindergarten, or transitional kindergarten), approximately 60% of children are not enrolled in any kind of preprimary school (NCES, 2022). For these children, their primary early learning environment is the home, where parents and caregivers—who may or may not have the knowledge needed to effectively support their children's early math learning—are their first teachers. Without more equitable beginnings, it is difficult to ensure that every child will have access to the best start in mathematics (Huinker et al., 2020).

THE HOME MATH ENVIRONMENT

The past decade has seen a growing number of studies focused on the critical importance of the Home Mathematics Environment (HME), and the important role that parents play in the earliest math learning of children. The quality of a learners' HMEs are crucial in preparing them for mathematics learning in school (Blevins-Knabe, 2016). Early exposure to math concepts in the home is often lacking, leading to disparities in readiness and achievement upon school entry. Factors influencing these disparities are thought to include socioeconomic status, parent education level, and parental engagement, with children from high-SES backgrounds typically having an advantage in early math skills (Blevins-Knabe, 2016). Meta-analyses reveal a wide variability in HMEs and confirm the critical role of parental influence on children's math readiness (Daucourt et al., 2021; Dunst et al., 2017), emphasizing the need to understand and enhance the HME—and parents' pivotal role in it—to improve children's preparedness for formal education.

PURPOSE OF STUDY

The purpose of this study was to examine parents' and caregivers' (referred collectively as "parents" here) perceptions around Mathematical Parenting, defined as the cognitions, utterances, behaviors, and practices of parents and caregivers that influence the development of their children's math understanding, knowledge, and skills. The study of the HME has several limitations that the present study seeks to address, including limited studies on parents' roles in the Home Math Environment (HME), along with a lack of standardized methodologies and tools for studying the HME (Blevins-Knabe, 2016). In response to this, the researchers devised the RESET framework which specifically seeks to examine how parents perceive their *role* in their child's math learning, their math learning *expectations* for their child, their perceptions of their own math *skills* and knowledge, as well as their parenting self-*efficacy*, and their perceived *time* and energy available to engage in supportive math parenting behaviors (see Table 1).

RESET Framework Domain Descriptions	
R (Role)	A parent's construction of their parental role is shaped by their early experiences in learning math, the ways in which they were parented, values and beliefs related to education, and other peer and societal or cultural influences. A parent's role is

	socially constructed and may change in response to changing social conditions, parent efforts and education (e.g., parent education or intervention programs, or even the child themselves), or the accumulation of life experiences.
E (Expectations)	The parent's expectations for the child's performance or development in mathematics is influenced by the value that the parent places on the learning of mathematics, its perceived role in the life of the parent and child, its perceived impact on the future success of the child, and the parent's knowledge and awareness of the mathematics concepts and skills appropriate for the child's age and developmental level.
S (Skills)	Parents' perceptions of their own mathematics skills and knowledge impact the ways in which they choose to interact with their children through mathematics, the types of skills and concepts they emphasize, and the expectations they have for their children's math development (e.g., if a parent feels like his or her life opportunities were limited because of weak math skills, they may conversely adopt higher expectations for their child's math learning in order to ensure the child is not limited by lack of math knowledge and skills).
E (Efficacy)	Parents' sense of self-efficacy is related to their belief in their ability to successfully support the math development of their child. It is influenced by their perceptions of math skills and knowledge, and influences their expectations for their child's math learning, as well as the ways in which they engage their child in mathematics activities.
T (Time)	Parent engagement in shared math activity is influenced by their perceptions of the time and energy available to participate. Further, parent perceptions of time and energy may be impacted by parents' perceived skills and knowledge and personal sense of self-efficacy. For example, more time and energy may be required from parents with low self-efficacy to engage meaningfully with their children through mathematics (more time to prepare in order to feel confident and comfortable, more anxiety that saps energy, etc.).

Table 1: The RESET Framework

THEORETICAL FRAMEWORK

The RESET Framework (Table 1) is grounded in Vygotsky's (1986) work detailing the importance of the more knowledgeable other in the learning dynamic, Bronfenbrenner's (1992) ecological systems theory of learning that describes how different spheres of influence shape an individual's (e.g., parent's) learning, beliefs, values, and behaviors, Mowder's (2005) parent development theory that describes the ways in which parents learn to do the work of parenting, and Hoover-Dempsey and Sandler's (1997) work on the factors that impact parental involvement in children's learning. Based on 847 parents' responses on the RESET survey, the current study reports parents' perceptions on *role, expectations, skills, efficacy, and time* in the HME.

METHODS

The research questions that guided this study are: (1) How do parents perceive their role, expectations, skills, efficacy, and time in their Home Early Mathematics Environments? (2) How do their perceptions on the RESET differ by race, gender, socio-economic status? The sample included a diverse, nationwide group of parents (*n = 847*) of 4-5-year-old children who had not yet started kindergarten. They were recruited and screened by a commercial panel provider (Innovate MR, n.d.) for participation in a digital online survey. The sample included parents who represented a diverse set of genders, ethnic, income, and educational backgrounds (see Table 2).

Category	Level	Frequency	Percent	Valid Percent	Cumulative Percent
Gender	Female	626	73.9	73.9	73.9
	Male	210	24.8	24.8	98.7
	Non-Cisgender	11	1.3	1.3	100.0
Ethnicity	White	596	70.4	70.4	70.4
	African American	83	9.8	9.8	80.2
	Hispanic or Latino	55	6.5	6.5	86.7
	Indigenous	12	1.4	1.4	88.1
	Asian	28	3.3	3.3	91.4
	Multiracial	73	8.6	8.6	100.0
Age	Under 25 years old	43	5.1	5.1	5.1
	25 to 34 years old	437	51.6	51.6	56.7
	35 – 44 years old	299	35.3	35.3	92.0
	45 and older	68	8.0	8.0	100.0
Income	Under \$50K Annual	391	46.2	46.2	46.2
	\$50K to \$100K Annual	299	35.3	35.3	81.5
	Over \$100K Annual	157	18.5	18.5	100.0
Education	HS/Less than college	313	37.0	37.0	37.0
	Some College	244	28.8	28.8	65.8
	Undergrad or Grad Degree	290	34.2	34.2	100.0
Child School	Daycare	36	4.3	4.3	4.3
	Preschool/Pre-k	514	60.7	60.7	64.9

Home	297	35.1	35.1	100.0

Table 2: Sample Demographics (n = 847)

RESULTS

Parents completed the RESET Survey Instrument (Betts & Son, 2022), which included 30 items, with 6 items each across the domains of RESET, which were previously tested in two separate studies and deemed reliable (α > .70). Parent responses on 6 items in each domain were combined to create composite mean scores for each domain (e.g., Role Mean of 5.67, etc.). These composite mean scores acted as proxies for parent perceptions of the RESET that describe the home early math environment, with higher composite means indicating more positive perceptions, and lower means indicating more negative perceptions. In general, parents expressed higher perceptions of Role (*M = 5.66, SD = .87*) and Expectations (*M = 5.76, SD = .84*), and lower perceptions of their math Skills (*M = 4.58, SD = 1.41*), math self-efficacy (*M = 5.09, SD = .99*), and Time that they devoted for their kid learning at home (*M = 4.76, SD = 1.06*) (see Table 3). However, means below 5.00 represent composite scores that indicate ambivalent or negative perceptions along a domain.

	Role (α = .72)	Expectations (α = .78)	Skills (α = .85)	Efficacy (α = .74)	Time (α = .73)
Mean	5.66	5.76	4.58	5.10	4.67
Std. Deviation	.87	.84	1.41	.99	1.06
Variance	.75	.71	2.00	.98	1.13

Table 3: Parent perceptions RESET (Means, SDs, Var.)

A series of statistical tests (e.g., One-Way ANOVAs, Tukey’s tests for multiple comparisons, Multiple Linear Regression, Two-Step Cluster Analysis, etc.) were performed to examine differences on RESET means across a number of categorical variables that include race, gender, socio-economic status (See Table 1). [put in some explainer for how the non-cisgender parents were groups with mothers or fathers]. Parent Gender revealed a statistically significant difference between Mothers and Fathers in their perceptions of Role (p = 0.012, 95% CI = [0.03, 0.36]), with Mothers having higher perceptions of their Role in their children’s math learning. Conversely, Fathers demonstrated significantly higher perceptions of their math Skills (p < 0.001, 95% CI = [0.36, 0.88]), and self-Efficacy (p = 0.0, 95% CI = [-0.01, 0.36]) than Mothers when engaging in mathematical parenting. There were no statistically significant differences in Expectation and Time between parents of either male or female gender, and parents who identified as non-cisgender.

Multiple linear regression analysis found that the independent variables of parent Age, Gender, Education, and Income were significant for one or more of the RESET domains. For the dependent variable of Role, parent age was found to be a significant predictor (*β = .086, p = .042*), while parent gender had a significant negative relationship (*β = -.140, p = .027*) confirming that mothers had significantly higher perceptions of their Role than did fathers. The linear regression equation for the dependent variable Role with both parent age and parent gender as predictors would be: Role = β0 + β1(parent age) - 0.140(parent gender (father)) + ε. For the dependent variable Expectations, parent age (*β = .112, p = .006*) and education (*β = .123, p = .002*) were significant predictors. For the dependent variable Skills, parent gender (*β = .331, p = .001*), education (*β = .228, p < .001*), and income (*β = .215, p = .003*) were significant predictors. For the dependent variable Efficacy, parent education was a significant predictor (*β = .127, p = .006*). Finally, for the dependent variable Time, parent age was a significant predictor (*β = .107, p = .040*). Accordingly, the following linear equations can be created:

- Expectations = β0 + 0.112(parent age) + 0.123(education) + ε;
- Skills = β0 + 0.331(parent gender) + 0.228(education) + 0.215(income) + ε;
- Efficacy = β0 + 0.127(education) + ε;
- Time = β0 + 0.107(parent age) + ε

These results suggest that parent age, gender, education, and income may have different impacts on different aspects of parents' perceptions of their math parenting roles and expectations. This was confirmed during two-step cluster analysis identified two distinct clusters including: Cluster 1, characterized by parents who tended to have less education (high school degree or no college), lower income (less than $50K annually), lower percentages of children in preschool-PreK, higher percentages of mothers, younger parents (25-35-years-old), and RESET means of Skills (M = 4.33), Efficacy (M = 4.98), and Role (5.66). Cluster 2 was characterized by parents who tended to have more education (college degree), higher income (over $100K annually), included higher percentages of children in preschool/PreK, fathers, older parents (35 years old and up), with RESET means of Skills (M = 5.06), Efficacy (M = 5.30), and Role (5.65). Comparing RESET means across the two clusters revealed that parents with lower education, income, higher percentage of mothers and children at home had much lower perceptions of their Skills and Efficacy than parents from Cluster 2, though both groups had similar perceptions of their Roles.

DISCUSSION

The research discussed here represents work in progress and has only begun to scratch the surface of how parents develop their perceptions around math parenting. Results show that different parent characteristics, such as gender, age, education level, and income all significantly impact parent perceptions around RESET, albeit in different ways. Results further suggest that for stakeholder partnerships with parents and families to be successful, different supports and resources should be considered for different subgroups of parents characterized by different needs. For example, younger parents (even of those with children in preschool or prekindergarten) may need more support than do older parents. Additionally, parents with less educational and income resources exhibit lower perceptions of their skills and efficacy and would also benefit from more support. Most of all, cluster analysis revealed that most children not enrolled in preschool or pre-K were being parented by parents with lower resources, skills, and efficacy (Cluster 1), making them most at risk for lack of math-learning readiness at the onset of schooling. Further research should consider how this subgroup of parents may be reached and better supported so that their children might also begin kindergarten ready to learn.

References

Betts, A. (2021). The RESET Framework: Examining critical factors in parent-child math participation. In IAFOR (Ed.). *The IAFOR International Conference on Education – Hawaii 2021 Official Conference Proceedings.* Paper presented at the IAFOR International Conference on Education: Independence and Interdependence, Hawaii. Japan: The International Academic Forum.

Betts, A., & Son, J-W. (2022). Why parents do what they do: Developing and validating a survey for the mathematical lives of parents and children. *The Paris Conference on Education 2022 Official Conference Proceedings.* Paper presented at the Paris Conference on Education. Paris, France: The International Academic Forum.

Blevins-Knabe, B. (2016). Early mathematical development: How the home environment matters. In Blevins-Knabe & Berghout Austin, A.M. (Eds.) *Early childhood mathematics skill development in the home environment*, (pp. 7-28). Springer International Publishing.

Bronfenbrenner, U. (1992). *Ecological systems theory*. Jessica Kingsley Publishers.

Claessens, A., & Engel, M. (2013). How important is where you start? Early mathematics knowledge and later school success. *Teachers College Record*, *115*(6), 1–29.

de Brey, C., Musu, L., McFarland, J., Wilkinson-Flicker, S., Diliberti, M., Zhang, A.,

Daucourt, M. C., Napoli, A. R., Quinn, J. M., Wood, S. G., & Hart, S. A. (2021, accepted). The Home Math Environment and Children's Math Achievement: A Meta-Analysis. *Psychological Bulletin.*

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., & Japel, C. (2007). School readiness and later achievement. *Developmental Psychology*, *43*(6), 1428–1446.

Dunst, C.J., Hamby, D.W., Wilkie, H., & Dunst, K.S. (2017). Meta-analysis of the relationship between home and family experiences and young children's early numeracy learning. In S. Phillipson, A. Gervasoni, & P. Sullivan (Eds.) *Engaging families as children's first mathematics educators: International perspectives* (pp. 105-125). Springer.

Hoover-Dempsey, K.V., & Sandler, H.M. (1997). Why do parents become involved in their children's education? *Review of Educational Research*, 67(1), 3-42.

Huinker, D., Yeh, C., & Marshall, A. M. (2020). Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations.

Innovate MR, (n.d.) *ESOMAR 28: Questions to help buyers of online samples.* InnovateMR. http://hubs.ly/H06gdg20

Mowder, B. A. (2005). Parent development theory: Understanding parents, parenting perceptions and parenting behaviors. *Journal of Early Childhood and Infant Psychology*, *1*, 45-65.

Nguyen, T., Watts, T.W., Duncan, G.J., Clements, D.H., Sarama, J.S., Wolfe, C., & Spitler, M.E. (2016). Which preschool mathematics competencies are most predictive of fifth grade achievement? *Early Childhood Research Quarterly, 36*, 550-560.

Ritchie, S. J., & Bates, T. C. (2013). Enduring links from childhood mathematics and reading achievement to adult socioeconomic status. *Psychological Science. 24*(7). 1301-1308.

Schweinhart, L., Montie, J., Xiang, Z., Barnett, W., Belfield, C., & Nores, M. (2005). *Lifetime Effects: The HighScope Perry Preschool Study Through Age 40*. In Monographs of the HighScope Educational Research Foundation, 14. HighScope Press

Vygotsky, L. (1986). *Thought and language: Newly revised and edited.* A. Kozulin (Ed). Cambridge, MA: The Massachusetts Institute of Technology.

FACILITATORS' CATEGORIES WHEN NOTICING A FICTIONAL PD-SITUATON: PEDAGOGICAL CONTENT VS. GENERAL PEDAGOGIC FOCUS

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Professional development (PD) courses contribute to the improvement of mathematics instruction, requiring facilitators to possess specific expertise for effective navigation of complex instructional scenarios. This study explores the identification of clusters pertaining to facilitators' categories when engaging in noticing of a fictional PD situation. Three clusters emerged: Cluster A exhibits the use of pedagogical content knowledge on the PD level (PCK-PD), cluster B accentuates general pedagogical content knowledge on PD level (GPK-PD), and cluster C strongly emphasizes GPK-PD. Although facilitators (N = 156) across clusters share similar teaching experiences, variations exist in their facilitation experiences. These findings provide valuable insights for aligning PD courses with the specific needs of facilitators.

INTRODUCTION

Professional development (PD) courses play a pivotal role in mathematics education, enhancing teachers' skills and contributing to the overall improvement of mathematics instruction (Prediger et al., 2022), with facilitators playing an important role in teacher learning (e.g. Borko et al., 2011). Specifically, facilitator expertise, including facilitators' ability to engage in noticing during PD courses, supports this learning. Drawing on van Es and Sherin's (2002) framework, noticing involves (a) recognizing the significance or noteworthy aspects of a situation, (b) establishing connections between specific interactions and the broader principles of teaching and learning they embody, and (c) employing contextual knowledge to reason about interactions within the given context. These skills are crucial for facilitator expertise in PD settings, enabling facilitators to adeptly navigate complex instructional scenarios. Research on facilitator expertise in mathematics education, addressing what facilitators need to know to create substantial, content-specific learning opportunities for teachers, emerged through the work of Zaslavsky and Leikin (1999). Recent research, however, has often emphasized generic aspects by viewing PD as a singular entity, thereby overlooking specific content-related considerations (Prediger et al., 2022). To consider content-related aspects, this study delves into aspects of content-related PD, exploring facilitators' categories, such as pedagogical content knowledge on the PD level (PCK-PD) and general pedagogical knowledge on the PD level (GPK-PD). We examine the categories facilitators use when engaging in noticing of a fictional PD situation, identify clusters and characterize them in terms of teaching and facilitation experience as a means of aligning PD courses with facilitators' needs.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 80–87). PME.

Facilitator expertise for noticing during PD courses

Van Es and Sherin's (2002) noticing framework for teachers can be transferred to facilitators and professional development settings. The three essential elements – identifying crucial aspects, making connections, and leveraging contextual knowledge for reasoning – apply across instructional and facilitative contexts (van Es & Sherin, 2002). This adaptability is evident when aligning this framework with the PID-model proposed by Kaiser et al. (2015), which emphasizes situation-specific skills such as perceiving events, interpreting activities, and decision-making. Whether in a classroom or a professional development setting, perception, interpretation, and decision-making are fundamental processes contributing to the diagnostic competence and adept navigation of diverse aspects of teaching or facilitation (Hoth et al., 2016).

While the examination of mathematics facilitators' expertise, especially for specific mathematics PD content, is still a developing field, research has shown how facilitators' expertise is essential for the quality of PD programs (Borko et al., 2011). Frameworks for considering teachers' expertise have been lifted to the facilitator level (Prediger et al., 2019), such as the ROGI framework that examines facilitators' resources, orientations, goals, and identities (Karsenty et al., 2023). Prediger et al. (2022) also lifted a framework for teacher expertise (Prediger, 2019) to a framework for content-related facilitator expertise, merging a cognitive and situated perspective (suggested, for example, by Borko et al., 2014). The framework contains jobs as typical and complex situational demands that go hand in hand with facilitation of specific PD content, and practices as recurring patterns of facilitators' utterances and actions for managing the jobs. Facilitators' practices can be determined by underlying categories, pedagogical tools, orientations, and situative goals.

For this paper, the knowledge that filters and focuses the facilitators' categories or categorial perception and thinking is tied in particular to their pedagogical content knowledge for teachers' professional development (PCK-PD). Thereby, facilitators' PCK-PD reflects the knowledge that facilitators bring to the PD concerning teachers' learning. This could include, for example, knowing about teachers' diagnosing of typical student challenges with the mathematical content and how to help teachers to best enhance their student learning with a conceptual focus, such that the construction of conceptual understanding of procedures and concepts is prioritized (e.g., Bray, 2011). Furthermore, PCK-PD includes addressing how to support teachers in focusing on long-term student learning by considering students' learning trajectories and connecting to their prior knowledge while also creating a cognitively activating and adaptive learning environment (Prediger et al., 2023). Another important aspect lies in enabling teachers to help their students engage in collaborative communication and discussion with one another about the mathematical ideas (Prediger et al., 2023; Walshaw & Anthony, 2008). General pedagogical knowledge on the professional development level (GPK-PD), on the other hand, includes managing and instructing courses and addressing challenges like teacher resistance and motivation, extending beyond mathematics (Prediger et al., 2022).

PRESENT STUDY AND RESEARCH QUESTIONS

Given the importance of facilitator expertise for the quality of PD programs, as highlighted by frameworks like ROGI (Karsenty et al., 2023) and Prediger's (2022) content-related facilitator expertise framework, our focus is on facilitators' categories, examining their ability to diagnose challenges of teacher learning, prioritize conceptual understanding, and help teachers create an engaging learning environment. To gain insight into where facilitators stand in terms of their noticing categories at the beginning of their own one-year PD-course, the following questions guided the study:

RQ1: Can different clusters for facilitators' categories be identified when they engage in noticing of a fictional PD situation?

RQ2: How can these clusters be characterized in terms of teaching and facilitator experience?

METHODOLOGY

Participants, context, and data collection

The present study is situated within the large-scale PD program *QuaMath*, which aims to improve PD quality, and, ultimately, mathematics instruction in Germany. The program duration is 10 years, with an outreach of approximately 10,000 schools from primary to secondary level. The focus on instructional quality is guided by five principles for mathematics teaching, gained by an extensive literature review: *conceptual focus*, *cognitive demand*, *student focus and adaptivity*, *longitudinal coherence*, and *enhanced communication* (Prediger et al., 2023). First, facilitators attend a one-year PD program dedicated to exploring the five principles in detail for different mathematics topics, and how to support teacher learning in this respect. Second, they then provide the PD courses for teachers themselves. Particularly, the facilitator PD aims to strengthen their situation-specific skills in relation to teachers' learning of the five principles.

PD facilitators from 15 out of 16 Federal States in Germany started the program in September 2023. Among them, N = 156 facilitators from the primary level, with M = 15.5 (SD = 9.7) years of mathematics teaching experience, opted to participate in this study. Facilitation experience ranges anywhere from 0 to more than 7 PD sessions held as facilitators. Prior to the start of the program, the facilitators completed a survey containing demographic questions as well as a situated assessment of their noticing skills in regard to teacher utterances regarding the main principles of the program. Thus, respective aspects could be raised with respect to noticing a fictional dialogue of teachers from a PD, discussing the use of a digital learning application (app) in the classroom (Figure 1). In this fictional dialogue, the teachers primarily discuss short-term and motivational benefits of the use of the learning app. Aspects such as teaching for conceptual understanding in a long-term manner, promoting cognitive activation for students, monitoring students' learning and grounding instructional decisions on their progress, as well as developing a communicative learning environment are not of focus. The facilitators were asked to respond to the following questions: 1) Briefly describe what stands out to you about this discussion amongst teachers in a PD. 2) How would you interpret the statements of the three teachers? 3) As a facilitator, how would you respond? Answers to questions 1 and 2 are viewed as the facilitators' perception and interpretation of the situation, with no further distinction being made between the two skills. Question 3 evokes facilitators' decision-making.

Situation in a mathematics PD:

Ms. Miller is raving about a new app that she used in her class:

Ms. Miller: The app is fantastic. My students can work exactly on their level, because the multiplication tasks will always be adjusted.

Ms. Özdemir: I know that app too. Then they just sit there in front of their tablets and practice multiplication. Although it is nice and quiet.

Mr. Meier: I also use it, everyone can work on his or her own level and just get the tasks that they need. And my students are really proud of themselves when they figure out lots of tasks and they have more confidence for the class test.

Figure 1: PD situation - dialogue amongst teachers concerning the use of an app.

Data analysis

Facilitators' responses to the three questions were analyzed regarding the categories they used to perceive, interpret and decide upon the situation. For the data analysis, we followed Prediger's et al. (2022) distinction between facilitators' general pedagogic knowledge (GPK-PD), and pedagogical content knowledge (PCK-PD) for teacher PD. Furthermore, we combined a deductive with an inductive approach. With respect to facilitators' PCK-PD, we coded the data with aspects of the five principles *conceptual focus*, *cognitive demand*, *student focus and adaptivity*, *longitudinal coherence*, and *enhanced communication*. For each answer, these categories were rated 0 (category was not addressed), and 1 (category was addressed). No additional categories emerged from the data. With respect to facilitators' GPK-PD, we drew on inductively coding facilitators' responses. In sum, we yielded the following six categories that were then finally coded for the whole data set: *atmospheric argumentation*, *general digital media focus*, *methodological individualization*, *short-term success*, *affective-motivational aspects*, and *general description*. For both PCK-PD and GPK-PD, each question response in its entirety comprised the unit of analysis. That is, the PCK-PD and GPK-PD subcategories were assigned to facilitators' perception/interpretation of the situation (questions 1 and 2), and their decision-making (question 3). A team of researchers piloted the coding system and after several rounds of discussion, one researcher compared the coding with those of the other members of the research team. Cohen's kappa between $\kappa = .88$ and $\kappa = .94$ for inter-rater reliability was achieved.

For the facilitators' perception/interpretation and decision-making, a sum code was built for the amount of codes of PCK-PD, and GPK-PD respectively. Subsequently, we conducted a hierarchical agglomerative cluster analysis, using Ward's method, in which similarly employed types of facilitators' categories were grouped together in an accumulating manner to generate clusters of facilitators using similar categories (Clatworthy et al., 2005). Levene's test was conducted to assess the homogeneity of variances across the clusters, and showed that equal variances for PCK-PD for perception/interpretation and decision-making, and GPK for decision-making could not be assumed for all of them ($p < .001$). To examine the statistical significance of differences of PCK-PD and GPK-PD usage in perception/interpretation and decision-making, we performed ANOVA and Welch-Test, which revealed significant differences between the clusters. Subsequent post-hoc tests (Tukey and Games Howell) revealed which clusters accounted for these differences.

RESULTS

In regard to the first research question, we yielded three different clusters for facilitators' categories when noticing the fictional PD situation. The first cluster, cluster A ($n = 45$), is characterized by facilitators relying rather on PCK-PD for both perception/interpretation (P/I) and decision-making (D). Thereby, PCK-PD entails mentioning aspects of *conceptual focus*, *cognitive demand*, *student focus and adaptivity*, *longitudinal coherence*, and *enhanced communication* in relation to the teacher utterances. Using PCK-PD for both perception/interpretation and decision-making is significantly higher in this cluster compared to the other two clusters (P/I: Welch's $F(2, 87.84) = 20.06$, $p < .001$; D: Welch's $F(2, 80.51) = 118.58$, $p < .001$). Cluster B ($n = 69$) exhibits a focus on GPK-PD in perception/interpretation and decision-making, accompanied by moderate scores on PCK-PD usage. GPK-PD comprises aspects such as *atmospheric argumentation*, *general digital media focus*, *methodological individualization*, *short-term success*, *affective-motivational aspects*, and *general description* to respond to teachers' statements in the fictional PD situation. Meanwhile, cluster C ($n = 42$) shows a strong focus on GPK-PD. The use of GPK-PD for both perception/interpretation and decision-making is significantly higher in cluster C than in the other two clusters (P/I: $F(2, 153) = 26.36$, $p < .001$; D: Welch's $F(2, 73.90) = 65.27$, $p < .001$). Table 1 provides an overview on the three clusters.

		Cluster A ($n = 45$) *PCK-PD focus*	Cluster B ($n = 69$) *GPK-PD focus*	Cluster C ($n = 42$) *strong GPK-PD focus*
PCK-PD	P/I	1.32 (0.98)	0.41 (0.58)	0.27 (0.47)
	D	2.20 (0.87)	0.13 (0.34)	0.14 (0.35)
GPK-PD	P/I	0.78 (0.80)	1.43 (0.90)	2.23 (1.09)
	D	0.69 (0.76)	0.97 (0.42)	2.33 (0.75)

Table 1: Clusters for facilitators' categories when noticing the fictional PD situation (means and standard deviation of PCK-PD, maximum score 5, and GPK-PD, maximum score 6).

The following statements from facilitators in cluster A exhibit what their focus on PCK-PD for both perception/interpretation and decision-making entails:

> I think that the teachers may not have internalized the principle of conceptual focus in math instruction or that they may see the app as a relief of their work, as so many children are at different performance levels. (Perception/interpretation, Facilitator A_1, PCK-PD subcategory: *conceptual focus*)

> I would ask what content goals are pursued with the app and which competencies can be promoted through it. Additionally, I would inquire about alternatives. What advantages does the app offer compared to other forms of instruction? (Decision-making, Facilitator A_2, PCK-PD subcategory: *longitudinal coherence*)

Instances from facilitator statements in cluster C, exemplify what a strong GPK-PD focus for both perception/interpretation and decision-making encompasses:

> Ms. Özdemir shows a rather negative attitude towards digital media. Ms. Maier keeps a close eye on her students regarding the positive effects this app can have. With every new idea, one should always consider how it can be integrated into the classroom and what benefits it has for teachers and students, as well as any potential drawbacks. (Perception/Interpretation, Facilitator C_1, GPK-PD subcategories: *general digital media focus, short-term success, general description*)

> I would appreciate and thank the participants for contributing their own experiences and prior knowledge. I would ask someone else to explain the app to the other participants and possibly facilitate its use if I am familiar with it. I would encourage all participants to share their own experiences with it. (Decision-making, Facilitator C_2, GPK-PD subcategories: *atmospheric argumentation, general digital media focus*)

Regarding research question 2, we further characterized the clusters in terms of facilitators' teaching and facilitation experiences (Table 2).

	Cluster A	Cluster B	Cluster C
Mathematics teaching qualification	yes: 88.9% no: 11.1%	yes: 82.6% no: 17.4%	yes: 88.1% no: 11.9%
Teaching experience (years)	13.90 (10.30)	16.30 (8.89)	15.73 (10.41)
Facilitation experience (PD courses held as facilitator)	0: 22.2% 1-6: 20.0% >7: 57.8%	0: 24.6% 1-6: 23.2% >7: 52.2%	0: 45.2% 1-6: 14.3% >7: 40.5%

Table 2: Facilitators' teaching and facilitation experiences for the three clusters.

The proportion of facilitators without a mathematics teaching qualification in clusters A and C is comparable, being higher in cluster B. Facilitators in cluster B possess the most teaching experience, while only slight differences exist among the clusters. In cluster A, with the PCK-PD focus, facilitators have the most experiences as facilitators, with 57.8% having held over 7 PD courses. Facilitators in cluster B show similar experiences, with 52.2% having facilitated over 7 PD sessions. In cluster B, 24.6%

have no facilitation experience, compared to 22.2% in cluster A and 45.2% in cluster C. Cluster C, whose facilitators showed a strong GPK-PD focus, comprises the cluster with the least facilitation experience.

DISCUSSION AND CONCLUSION

With respect to facilitators' situation-specific skills, we found differences regarding the categories that were applied when perceiving and interpreting a fictive PD situation and making decisions on how to deal with teachers' statements. The three clusters reveal substantial differences of how facilitators noticed the situation, ranging from a PCK-PD to a strong GPK-PD focus. Facilitators who applied PCK-PD categories discussed, for instance, whether teachers questioned if the app helps students to gain conceptual knowledge or if long-term goals can be achieved. On the contrary, facilitators from the other two clusters focused on GPK-PD categories, placing emphasis, for example, on atmospheric aspects, such as appreciating teachers' statements. Alternatively, they highlighted that teachers attended to students' sense of achievement by the short-term success provided by the app. Even more so, facilitators in cluster C applied many different GPK-PD aspects, and did not rely on content-specific aspects of teachers' learning. In sum, we gained insight into how differently facilitators perceived, interpreted and decided how to react to the fictional PD situation. The three clusters also revealed differences regarding facilitators' background. They share almost the same amount of teaching experience, with nearly 20% of facilitators in cluster B teaching out-of-field. However, they differ from each other regarding their facilitation experiences. In cluster A, which has a focus on PCK-PD, facilitators are the most experienced, with experience decreasing from cluster B to C. In cluster C, nearly 50% of the facilitators are inexperienced.

In our study, we accessed facilitators' approximated noticing by using a situated approach that allows for economically gaining insight into their situation-specific skills. One limitation lies in using only one situation and analyzing categories as only one specific aspect of facilitators' expertise. However, we gained insight into where facilitators stand in view of their noticing categories at the beginning of their own one-year PD course. Within the *QuaMath* project, facilitators' gaining of PCK-PD categories for supporting teachers' learning of the five principles for substantial mathematics teaching is in the focus. So far, such PCK-PD aspects are only evident for facilitators in one cluster. Our research thus contributes relevant information on how to align PD design to fit facilitators' specific needs.

Additional information

This research was funded within the project *QuaMath*, financed by the Standing Conference of the Ministers of Education and Cultural Affairs (KMK) in Germany.

References

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice-based professional development programs. *ZDM Mathematics Education, 43*(1), 175-187. https://doi.org/10.1007/s11858-010-0302-5

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *Journal of Mathematical Behavior, 33*, 149–167.

Bray, W. S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education*, *42*(1), 2–38. https://doi.org/10.5951/jresematheduc.42.1.0002

Clatworthy, J., Buick, D., Hankins, M., Weinman, J., & Horne, R. (2005). The use and reporting of cluster analysis in health psychology: A review. *British Journal of Health Psychology, 10*(3), 329–358

Hoth, J., Döhrmann, M., Kaiser, G., Busse, A., König, J., & Blömeke, S. (2016). Diagnostic competence of primary school mathematics teachers during classroom situations. *ZDM Mathematics Education, 48*(1), 41–53.

Kaiser, G., Busse, A., Hoth, J., König, J., & Blömeke, S. (2015). About the complexities of video-based assessments: Theoretical and methodological approaches to overcoming shortcomings of research on teachers' competence. *International Journal of Science and Mathematics Education, 13*(2), 369–387.

Karsenty, R., Pöhler, B., Schwarts, G., Prediger, S. & Arcavi, A. (2023). Processes of decision-making by mathematics PD facilitators: the role of resources, orientations, goals, and identities. *Journal of Mathematics Teacher Education*, *26*(1), 27-51. https://doi.org/10.1007/s10857-021-09518-z

Prediger, S. (2019). Promoting and investigating teachers' pathways towards expertise for language-responsive mathematics teaching. *Mathematics Education Research Journal, 31*, 367–392. https://doi.org/10.1007/s13394-019-00258-1

Prediger, S., Roesken-Winter, B., Stahnke, R., & Pöhler, B. (2022). Conceptualizing content-related PD facilitator expertise. *Journal of Mathematics Teacher Education, 25*, 403–428. https://doi.org/10.1007/s10857-021-09497-1

Prediger, S., Dröse, J., Stahnke, R., & Ademmer, C. (2023). Teacher expertise for fostering at-risk students' understanding of basic concepts: Conceptual model and evidence for growth. *Journal of Mathematics Teacher Education, 26,* 1–28. https://doi.org/10.1007/s10857-022-09538-3

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10*(4), 571–596.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research, 78*(3), 516–551. https://doi.org/10.3102/0034654308320292

Zaslavsky, O. & Leikin, R. (1999). Interweaving the training of mathematics teacher educators and the professional development of mathematics teachers. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 143–158). PME.

STUDENTS' EXPLANATIONS FOR UNIT CONVERSIONS: SPECIFYING UNDERLYING STRUCTURES TO BE ADRESSED

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Visual models have been widely used to promote students' understanding for mathematical procedures. Yet students' explanations using visual models can reveal underestimated complexities that need to be unpacked to provide targeted learning opportunities. In our qualitative study, we analyzed how 10–12-year-old students explain the conversion of mass units, and we unpacked what different connections between representations need to be verbalized. The analysis revealed that students who connect the representations draw upon three kinds of underlying structures: bundle structures, refinement structures, and place-value structures. All should be explicitly focused on and supported in future designs for teaching-learning arrangements.

EXPLAIN UNIT CONVERSION: A LEARNING GOAL TO BE UNPACKED

Increasing consensus exists that students should not only learn to explain the meaning of mathematical *concepts* (e.g., fractions), but also *procedures* (e.g., multiplication of fractions; Kilpatrick et al., 2001), often by support of visual models. For various arithmetical procedures, students have been engaged in explaining procedures through visual models, and studies have repeatedly shown that visuals do not automatically help unless students focus on the relevant mathematical structures in the visuals (Fuson et al., 1997; Glade & Prediger, 2017). Based on these findings, learning environments for arithmetical procedures have been increasingly focused on relevant structures.

Our overall design research study aimed at transferring this approach from arithmetic to an essential procedure in measurement, *unit conversion*, which can also be conducted symbolically or by visual-based strategies (e.g., with the visual model in Figure 1). In this paper, we qualitatively analyze how students explain the conversion of mass units with the visual model to unpack what its explanation entails in detail.

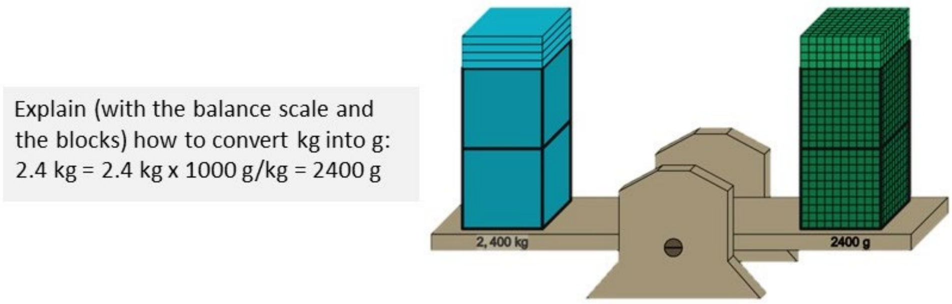

Figure 1: Explaining the procedure of converting mass units: task and design of a visual tool (Bielinski et al., submitted)

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 88–95). PME.

THEORETICAL BACKGROUND

Explaining with visual models requires focus on underlying structures

For enhancing students' understanding of procedures, visual models (graphical representations or manipulatives) have been widely used, particularly in arithmetic education (Lesh, 1979). Yet students have been shown not to profit automatically from the use of visuals unless they learn to impose structures onto the visuals, for instance, in the part-whole structures for adding in fraction bars (Lesh, 1979), place-value structures onto base-ten blocks for multi-digit subtraction (Fuson et al., 1997), or multiplicative bundle structures for distributive factorizing of multiplications (Clark & Kamii, 1996; Tondorf & Prediger, 2022).

In particular, when students are asked to explain why a procedure works, the explicit description of *structures* is crucial (Glade & Prediger, 2017). Hence, instructional support should be provided for students to focus on and articulate those conceptual structures underlying the procedure in view (Fuson et al., 1997). A typical schematization learning trajectory can thereby be designed in which students learn to (1) translate mathematical objects and operations between symbolic and visual representations, (2) enact visual-based strategies to represent the procedure in view and read off the result, (3) internalize the mathematical structures underlying the visual-based strategies so that the strategies can be pursued using only mental processes, (4) find patterns in informal visual-based strategies to schematize them into a formal procedure, and (5) justify the formal procedure by the underlying structures and patterns (Treffers, 1987; Glade & Prediger, 2017). Whereas Steps 1 and 2 of the learning trajectory can be conducted in purely empirical modes of enactment and reading off (perhaps without seeing underlying structures), Steps 3–5 can be promoted by asking students to explain the procedures and focusing their attention on the underlying structures.

To support students' focus on structures within the design of the learning environment, however, the most relevant structures for a particular topic must be specified by the design researcher, so that adequate scaffolds can be constructed (Fuson et al., 1997).

Transferring the schematization approach to a less investigated topic: Explaining procedures of mass unit conversion

Compared to the well-researched area of arithmetic, measurement has continued to be "a key, but under-researched area of the learning of mathematics" (Cheeseman et al., 2017, p. 144). While students' understanding of key measurement concepts (such as length or area) has been thoroughly investigated, little is known on students' understanding of *unit conversion*, an essential procedure in the field of measurement (summarized in Smith & Barrett, 2017). For our study, we chose mass, the most neglected among the measurement quantities (Cheeseman et al., 2017; Smith & Barrett, 2017), aiming to unpack what exactly is needed to explain how to convert kilograms into grams. For this, we aimed at transferring schematization pathways established in arithmetical procedures (see above) to unit conversion for mass units.

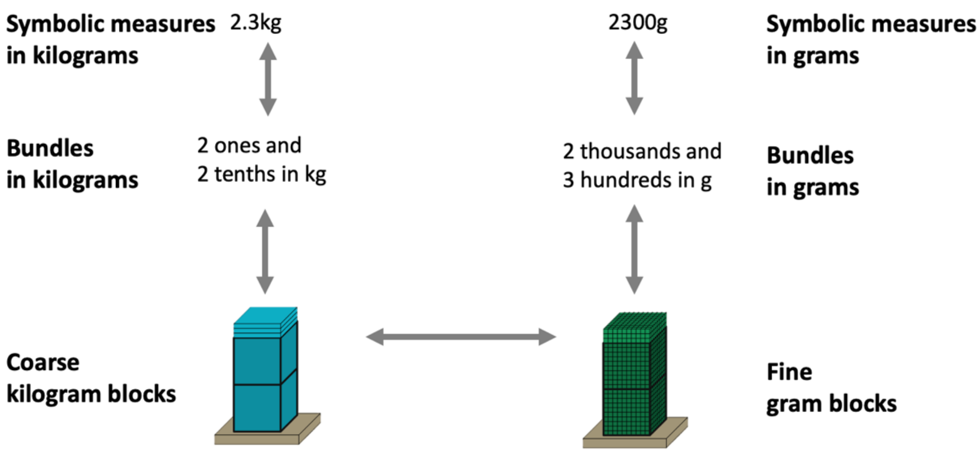

Figure 2: Conceptual framework for initiating and analyzing student explanations in visual-based conversion of mass units

In the digital teaching learning arrangement that we have designed in our overarching design research project *divomath* (Bielinski et al., submitted), students are introduced to the visual model of base-ten blocks on the balance scale (Figure 1). The balance scales conveys the relational meaning of the equal sign as "equally heavy," the base-ten blocks support bundle structures of 1000 in each block and base 10 in each digit.

Students' learning opportunities are sequenced in a schematization trajectory similar to those described above for arithmetical procedures in the following steps: Students first learn to interpret and compare mass units through translating symbolic measures into blocks ("2400g can be represented by two 1000g blocks and four 100g flats." "2.4kg can be represented by two 1kg-blocks and four flats of 0.1kg."). The conversion equality "2.4kg = 2400g" is then interpreted as an equilibrium on the balance scale that can be read off the interactive digital balance scale tool *in an empirical mode* without explicit reference to structures (the balance scale goes down for the heavier blocks). To *explain* the unit conversion (and internalizing structures), students need to overcome this empirical mode of reading off the visual and refer to the bundle structures underlying the place-value system and the conversion factor 1000: For converting 2400g into kg, students need to *connect the symbolic representation* of "2400g" with the *bundle representation* "2 thousands and 4 hundreds in g" and then to the *visual representation* of the two large green blocks and four green flats. These green blocks and flats are as heavy as two blue blocks and two blue flats, so the student converts this by exchanging green for blue blocks within the *visual representation*. Before translating this representation into symbolic measures in kg, students need to connect them to the *bundle representation*: two ones and four tenths in kg, which can then be translated into the *symbolic representation* as 2.4kg. Once the students understand that the blocks with equal weight come with refining a 1kg block into a 1000g block, they can schematize the visual-based conversion strategy into a symbolic transformation of 2400g x 1000 g/kg = 2.4kg. Hence, the bundle structure is a representation mediating between the symbolic and visual representation (as in Fuson et al., 1979), further structures can be identified empirically.

Research question

Connecting six different representations bears a remarkable complexity. The state of research on similar schematization challenges for arithmetical procedures (Fuson et al., 1997; Tondorf & Prediger, 2022) has suggested that students might produce heterogenous ways of explaining the connections. We unpack these complexities and heterogeneities in two research questions:

RQ1. How do students explain unit conversion of mass units using the visual model?

RQ2. On which underlying structures do they draw to connect representations?

METHODOLOGICAL FRAMEWORK

Methods of data collection for unpacking students' oral explanation

The study presented in this paper is embedded in a larger design research study focusing the reconstructive part rather than the iterative design consequences. As 10–12-year-old children rarely write long explanations, we captured their oral explanations articulated as they worked with the interactive visual model (Figure 1). For this, we conducted design experiments in laboratory settings with 18 pairs of students, participating in two or three 45-minute design experiment sessions each. From about 34 hours of video recorded during design experiment sessions on mass unit conversion, we transcribed selected episodes and extracted students' explanations.

Methods of data analysis

For the qualitive analysis of students' transcribed explanations for the focus task from Figure 1, we applied a deductive-inductive coding procedure: In Step 1, we deductively started from a coding scheme developed for visual-based strategies for transforming expressions (Tondorf & Prediger, 2022) and adapted it for visual-based strategies for unit conversion. With the adapted coding scheme, students' utterances were coded with respect to (a) the explicitly addressed representations (six vertices in Figure 2) and (b) the kind of connections they articulated between the representations (edges in Figure 2). In Step 2, the coding scheme was inductively deepened to infer the underlying structures on which students drew when articulating connections. The analytic outcomes are presented diagrammatically, with six examples shown in Figure 3. Non-addressed representations are marked by grey vertices, and addressed representations with full colors. Explicitly articulated connections are drawn using arrows on the edges. Parts of the transcript are shown in the vertices and edges of the diagram.

EMPIRICAL INSIGHTS

Figure 3 collects four examples selected as typical for the whole spectrum of explanations that 36 students provided in design experiments and their analysis. The four explanations explicitly connect different representations (non-articulated representations are marked in grey). Jens stays within the representations for kilograms and hints to grams only vaguely (Turn 1c), whereas the other three explanations address symbolic and visual representations for kilograms and grams.

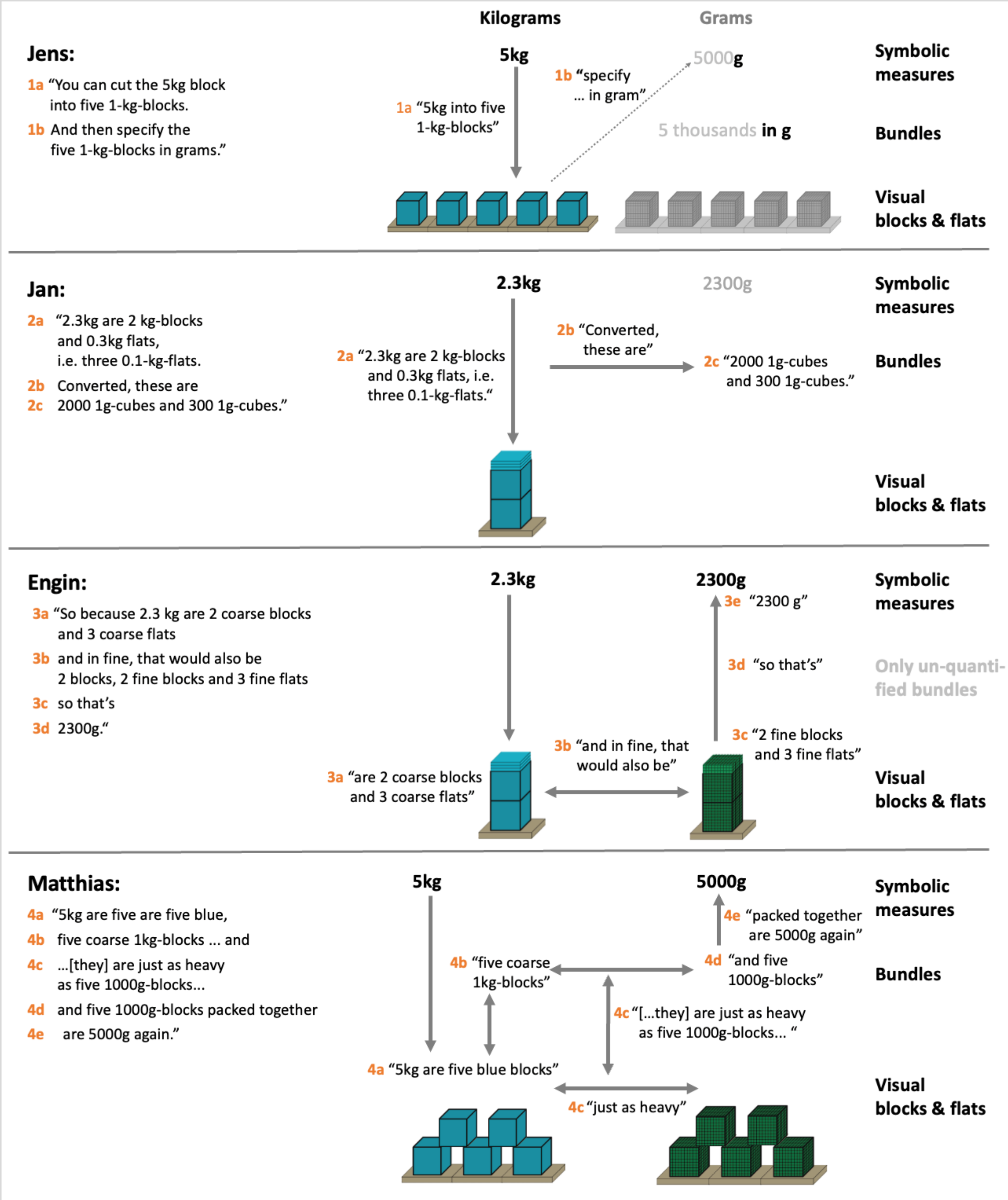

Figure 3: Four explanations and their analytic schemes for addressed representations and articulated connections

Among them, Jan and Matthias explicated the bundle representation that they expressed as directly tied with the visual representation (not in an abstract form as "the thousands" but "the kg-block," Turn 2a). Engin addressed the coarse and fine blocks for the large and small units only without quantification by talking about "in fine"

(Turn 3b), yet not about the base 1000. We interpret "it would also be" (Turn 3b) as a unidirected expression of equilibrium. Behind these heterogenous connections, we can identify underlying structures to which students explicitly or implicitly refer:

Jens uses *bundle structures* without alluding to place values when he "cuts a 5kg block into five 1kg-blocks" (Turn 1a), with a remarkable concise language, he explicitly distinguished the large bundle of the "5kg" from smaller bundles "1kg-block" which can be counted as 1, 2, 3, 4, 5 1kg-blocks, which is the core idea of multiplicative thinking. Similarly, Jan treated tenths as bundles: "0.3kg flat, i.e., three 0.1kg-flats" (Turn 2a). It is interesting to see that within our four examples here, the bundle structure was always seen in the visual representation, an important indication that the visual model served its first purpose. Other students (whose explanations are not shown here) also expressed bundles independently from the blocks.

The bundle structures were often embedded in uses of other features of *place-value structures:* Jan additively split the 2.3kg into ones and tenths (Fuson et al., 1997), saying "2.3kg are 2 kg-blocks and 0.3kg flats" (Turn 2a) and interpreted the face value "3" multiplicatively as "three 0.1kg-flats" (Turn 2a). Similarly, Matthias "packed together" (Turn 4e) five 1000g-blocks into 5000g, using the multiplicative characteristic for turning face values into place values. Multiplicative bundle structures are an integral part of the place-value system, and then need to be combined with other place-value features, as the role of place and face values and additive splits.

The third relevant kind of structures are *refinement structures*. Refinement structures can be addressed qualitatively as Engin's explanation of "two coarse blocks" (Turn 3a) that later became "two fine blocks" (Turn 3b). He articulated his awareness of refinement structures by saying, "and in fine, this would be also two blocks" (Turn 3b). Jan expressed refinement structures when "converting" (Turn 2b) 2kg blocks into the unbundled "2000g cubes" (Turn 2c), using refinement and bundle structures in the same moment. Matthias related the two visual representations in an empirical approach, by "just as heavy" (Turn 4c). But he immediately combined his empirical approach with making refinement structures explicit, not for the blocks alone, but also for the bundle structures: "five coarse 1kg blocks ... are just as heavy as five 1000g-blocks" (Turn 4b-d) is the most interesting phrase. It differs from Engin's purely qualitative articulation of the refinement structures in that it also mentions the bundle of 1000. We depicted this other kind of de-bundling with a vertical line connecting two horizontal lines in the analytic scheme (Figure 3). In total, we see four different ways of expression the conversion process: in a formal symbolic language (Turn 1b); related to the visually represented balance scale in an empirical approach, embedded in a complex complete phrase (Turn 4b-d) in which the refinement of the bundles is expressed, yet not completely explicit; in finer and coarser structures in comparisons (Turn 3b); and in visual-based conversions (Turn 2b).

DISCUSSION AND OUTLOOK

Heterogeneity of addressed connections and underlying structures

Explaining procedures by means of visual models is generally challenging for children (Fuson et al., 1997; Glade & Prediger, 2017; Tondorf & Prediger, 2022); this also applies to the conversion of mass units in our study. The analysis revealed that children were nevertheless able to succeed to manage an amazing complexity. The four explicitly presented cases span an impressive heterogeneity in terms of (a) explicitly addressed representations (all six are addressed overall, but not each by everyone), (b) explicitly articulating how the representations are connected (all seven potential connections are articulated by at least one student, but in very different ways), and (c) drawing upon different underlying structures. These findings resonate with those in the complete data set of 36 students. Figure 4 summarizes the structures we identified as relevant in (c) for (a) and (b): besides the *bundle structure* of multiplicative thinking (Clark & Kamii, 1996), we identified *base-ten place-value structures* with additive splits and place-/face-value features (Fuson et al., 1997) and *refinement structures* (Glade & Prediger, 2017) as essential for explaining unit conversion of mass units. For the 36 children of the age group in view (10-12 year old), the refinement structures seemed to be the newest and most challenging to grasp.

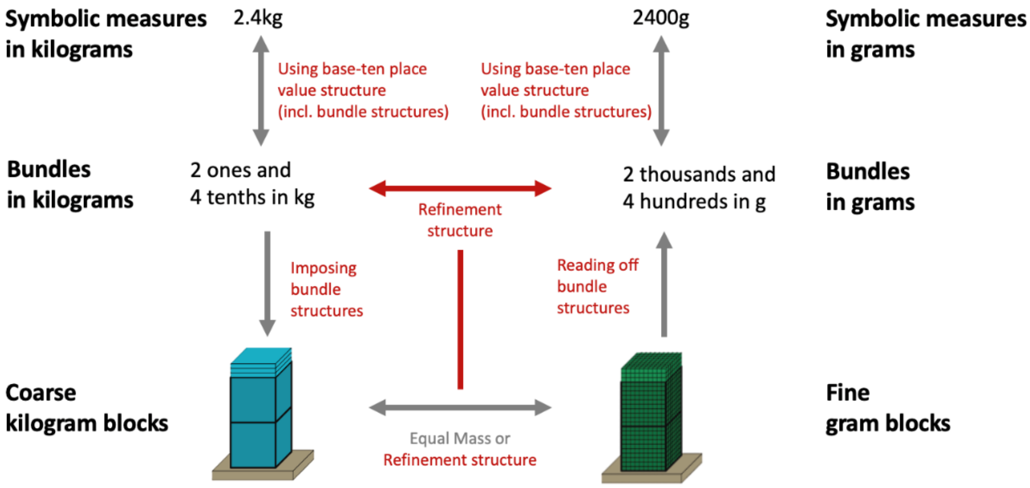

Figure 4: Empirical findings (new in red): Structures identified as relevant for explaining unit conversion through connecting representations

Limitations and outlook

Of course, the empirical findings must be interpreted without hasty generalizations, as the sample size of 36 students does not allow statistical generalizability. More importantly, the findings are strongly tied to the visual model we offered and the tasks and prompts by which students were guided to deal with it. Future studies should investigate the transferability of the findings to other learning environments, other visual tools, and other measurement quantities such as length, time, and money.

The current findings, however, already substantially inform the redesign of the teaching learning arrangement: As students' focus on underlying structures is so critical, scaffolds for focusing students' attention were successively integrated, as were language learning opportunities for articulating the structures. The explanation of Matthias in Figure 3 stems from Design Research Cycle 3 in which these integrations had already taken place. Further analysis and also a controlled trial are planned to validate our current assumption that these structure-focusing phrases can indeed scaffold students' processes of understanding and explain the connections.

Funding. The design research project *divomath* is financially supported by the State Ministries of Education of North Rhine-Westphalia and Brandenburg (grant to S. Prediger/C. Selter). We thank our project partners.

References

Bielinski, S., Peters, N., & Prediger, S. (submitted). Digital gestützte Lernpfade hin zum verständigen Umgang mit Gewichten. Submitted to F. Dilling, D. Thurm & I. Witzke (Eds.), *Digitaler Mathematikunterricht in Forschung und Praxis.* WTM.

Cheeseman, J., McDonough, A., & Golemac, D. (2017). Investigating children's thinking about suspended balances. *New Zealand Journal of Educational Studies*, *52*(1), 143–158. https://doi.org/10.1007/s40841-016-0073-9

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education*, *27*(1), 41–51. http://www.jstor.com/stable/749196

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education*, *28*(2), 130–162. https://doi.org/10.2307/749759

Glade, M., & Prediger, S. (2017). Students' individual schematization pathways. *Educational Studies in Mathematics*, *94*(2), 185-203. https://doi.org/10.1007/s10649-016-9716-5

Kilpatrick, J., Swafford, J., & Findel, B. (2001). *Adding it up. Helping children learn mathematics*. National Academy Press.

Lesh, R. (1979). Mathematical learning disabilities. In R. Lesh, D. Mierkiewicz, & M. Kantowski (Eds.), *Applied Mathematical Problem Solving* (pp. 111–180). Ericismeac.

Smith, J. P., & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355–385). National Council of Teachers of Mathematics.

Tondorf, A., & Prediger, S. (2022). Connecting characterizations of equivalence. *Educational Studies in Mathematics.*, *111*(3), 399–422. https://doi.org/10.1007/s10649-022-10158-0

Treffers, A. (1987). Integrated column arithmetic according to progressive schematisation. *Educational Studies in Mathematics*, *18*(2), 125–145. https://doi.org/10.1007/BF00314723

DO YOU SEE MATH? HOW BAYESIAN INFERENCE AND INTERNET MEMES CAN SHED LIGHT ON STUDENTS' UNDERSTANDING OF MATHEMATICAL CONCEPTS

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This study examines mathematics students' engagement with visual resources, using Bayesian inference and Wittgenstein's "seeing as" concept to explore how they use meme templates to represent mathematical concepts. The analysis of memes created by two high-school students reveals the significant influence of their mathematical knowledge and conceptual understanding on their representations, uncovering strengths and weaknesses not easily captured by conventional tasks. The findings contribute a fresh viewpoint on students' understanding of mathematical concepts, broadening the conversation on the role of visual inputs in mathematics education.

INTRODUCTION: THE IMPORTANCE OF VISUAL INPUTS

The role of sensory perception in the construction of mathematical concepts is widely considered pivotal (Andrá et al., 2015; Arzarello et al., 2005). Among sensory perceptions, visual inputs are deemed crucial for the understanding of mathematics (Arcavi, 2003; Duval, 1999; Presmeg, 2006; Radford, 2010). This importance stems not only from the fact that the ability to process visual representations is considered a fundamental aspect of human cultural development since cave paintings (Cecchinato, 2009) but also from the "pictorial turn" of 21st-century culture (Mitchell, 1995, p. 15). This shift has overturned the long domination of written text in Western scholarly culture. Historically, images were considered a means to communicate with people with limited literacy, while written texts were the prerogative of a cultured elite. Fuelled by technology, facilitating the creation and diffusion of visual resources, the *pictorial turn* has promoted images to the centre of contemporary "communication and meaning-making" (Felten, 2008, p. 60).

THEORETICAL FRAMEWORK: *SEEING* AND *SEEING AS*

While recognizing the significance of visual inputs in aiding communication and meaning-making in the mathematics classroom, the task of choosing appropriate visual resources to effectively support learners in constructing and understanding mathematical concepts is challenging. Cognitive science has shown that the way our brain organises, identifies, and interprets visual sensorial stimuli is subjective and strongly influenced by our previous knowledge and expectations (Bernstein, 2008; Seriès & Seitz, 2013). Indeed, research has shown that the processing of visual information involves both a *bottom-up* and a *top-down* progression. In the *bottom-up* phase, our brain processes inputs piecing them together to build up higher-level information (e.g., shapes for object recognition). In the *top-down* phase, our

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 96–103). PME.

expectations (informed by our prior knowledge) influence our perception. These two phases correspond to what Wittgenstein in his *Philosophical Investigations* (1953) calls *seeing* (the *bottom-up* input processing) and *seeing as* (the *top-down* personal interpretation). Wittgenstein illustrates his concept of *seeing as*, emphasizing the subjective nature of interpretation in visual experiences, through the study of ambiguous images, exemplified by the duck-rabbit illusion in Figure 1.

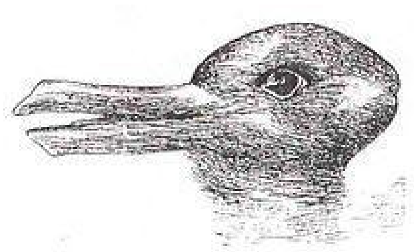

Figure 1: The duck-rabbit illusion

This visual puzzle, first published in the German magazine Fliegende Blätter on October 23rd, 1892, and subsequently investigated by the American psychologist Joseph Jastrow in 1899, presents an image that can be interpreted as either a duck or a rabbit, depending on the viewer's perception. Since its first appearance, the image has become paradigmatic to highlight how "we see with the mind as well as the eye" (Kihlstrom, n.d., par. 2), shedding light on the malleability and subjectivity inherent in our perceptual experiences. Indeed, it shows that perception is shaped not only by external stimuli but also by mental processes and expectations.

The influence of our expectations on the processing of visual stimuli is exemplified by Brugger and Brugger's study (1993). The study shows how a sample of people presented with a stylised version of the duck-rabbit illusion tended to see it as a rabbit around Easter and as a duck (or a similar bird) in October. The testing place was at the main entrance of the Zurich Zoo in Switzerland; thus, we can imagine that the subjects' expectations could be curved towards seeing an animal, but why a rabbit at Easter and a duck in October? We note that in Switzerland rabbit images at Easter and ducks in autumn are part of a shared folklore. Thus, Brugger and Brugger show us not only how much the subjects' expectations shape their *seeing as* but also that, through this *seeing as*, we gain knowledge about the culture that infuses these expectations.

Research in cognitive neuroscience models the process of *seeing as* using Bayesian inference (Seriès & Seitz, 2013), which allows inferring how our brain evaluates the conditional probability that an interpretative hypothesis is true given some input data, taking into account a prior probability represented by existing knowledge. Specifically, if in Bayes formula $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$ we consider A (Hypothesis) as the subject's interpretative hypothesis of some visual sensory inputs B (Data), then Bayesian inference offers a mathematical model of how our cognitive system evaluates the reliability of an interpretation of visual sensory inputs P(Hypothesis|Data), based on the *Likelihood* between data and hypothesis P(Data|Hypothesis), on *Prior* expectations P(Hypothesis), and on the *Clarity* of data P(Data), as detailed in Figure 2.

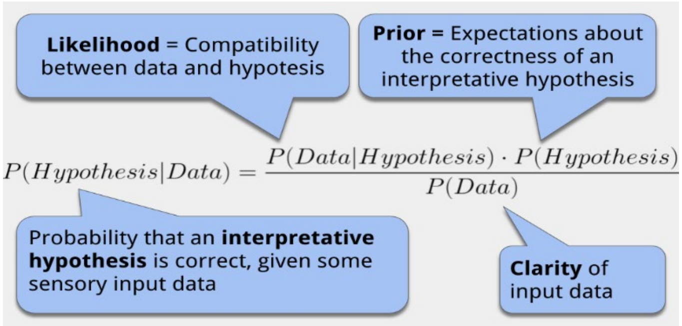

Figure 2: Bayesian inference on visual inputs interpretations

Thus, the subject's perceived probability that an interpretative hypothesis of some sensory input data is correct depends not only on the compatibility between data and hypothesis (Likelihood) but also on the ratio Prior/Clarity between the subject's expectations and the explicitness of input data. Focusing on this ratio, we see that its value gets bigger either when the Prior is very strong, or when the Clarity of the input data is very small. Thus, the more visually ambiguous the input data, the more the Prior influences the interpretation, as happens in Brugger and Brugger (1993). To sum up, our culture and existing knowledge constitute a Prior through which we interpret our visual perceptions, and their impact and visibility are all the more evident the vaguer is the information contained in the perceived visual stimuli (Esposito et al., 2023).

INTERNET MEME TEMPLATES AS VISUALLY AMBIGUOUS STIMULI

Coming to the teaching and learning of mathematics, we can imagine gaining knowledge about students' *Mathematical Prior*, i.e., their knowledge and understanding of mathematical concepts, by investigating how they mathematically interpret images that can be considered visually ambiguous inputs from a mathematical standpoint. Examples of these images are meme templates (Figure 3): popular images diffused on the Web, that users utilise in creating Internet memes.

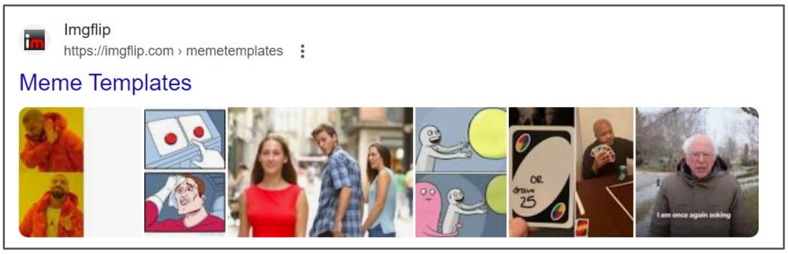

Figure 3: Samples of meme templates (source Imgflip)

Internet memes are a form of online cultural expression that rapidly spreads across the Web. They are typically relatable images accompanied by user-generated captions that evolve as users creatively modify and share them. Memes typically reflect current events, social trends or experiences, providing a shared language and cultural references within online communities (Bini et al., 2022). Meme templates are the foundational elements from which various mutations of Internet memes are created. They serve as a framework for users to insert their own content while adhering to the established format (Bini et al., 2023). Templates contribute to the virality of memes by offering a familiar structure that users can adapt to convey their unique perspectives or reactions, overlaying original texts to express different messages, and fostering a sense of collective humour and connection in the landscape of digital culture.

Meme templates are born outside the mathematical context and are visually ambiguous from a mathematical standpoint, i.e. they do not have an intrinsic mathematical content. Nevertheless, they can be *seen as* representations of mathematical concepts and, in this sense, they are used within dedicated online communities to create memes representing mathematical statements (Bini et al., 2022).

This research is therefore guided by the following research question: What prior mathematical knowledge and understanding can we infer from students' seeing meme templates as representations of mathematical concepts?

METHODOLOGY AND METHODS

Data come from a school experiment conducted in May 2018 with a group of 27 12th-grade learners in a scientific-oriented high-school in Italy. The task, to be completed individually at home as an end-of-the-year recap activity, was to create a mathematical meme on one of the year's maths topics and record an explanatory video of the addressed mathematical concept. Students shared memes and videos through a collective digital space, using the free Web-app Padlet, and subsequently their productions were the focus of a class discussion.

Assigning this task sets a cultural scenario that we can expect to activate students' mathematical priors influencing the act of *seeing as* meme templates as representations of mathematical concepts. This cultural scenario is in turn influenced by the way of schooling mathematics in the geographical environment that constitutes the setting for the experiment.

In the following July 2018, two of the students were interviewed, Mario and Luca (pseudonyms), selected because they created memes that were particularly interesting as they used templates to carry part of the mathematical meaning and not simply to emotionally reinforce the meme's message. Students were interviewed together, and they were both asked to explain how they got their idea for the meme. Interviews have been audio-video recorded, transcribed and subsequently analysed utilising a qualitative methodology and adopting an interpretative approach as outlined by Cohen et al. (2007), focussed on eliciting how Mario and Luca's mathematical priors influenced their view of meme templates as representations of mathematical concepts.

DATA AND ANALYSIS

Mario's mathematical meme

Mario's meme (Figure 4, centre) focusses on the study of a function, specifically on the process of plotting the first derivative sign chart to find a function's local maximum. In the cultural context where the experiment was conducted, it is customary to complete this sign chart with slanted arrows, pointed upward for increasing intervals and downward for decreasing ones. Body shapes in the template used by Mario (Figure 4, left) evoke the outline of these slanted arrows, that also appear in his explanatory video (Figure 4, right).

Researcher: How did you get your idea for the meme?

Mario: I was scrolling through various images [on Instagram] and I came across this picture [Figure 4, left], obviously without text, with these three people put in strange positions and the mind of a normal person says *ah that's nice, they are in strange positions*, while the mind of a math student says *this could represent a function* and therefore be effectively seen as a function following the lines of the bodies and the vertices formed by the heads and feet look like the maximum and the minimum of the function, so I said it could be a creative way to find the maximum and minimum of a function

Researcher: Ok, so it was the image that evoked a mathematical thought […] while scrolling, I get a mathematical idea

Mario: exactly!

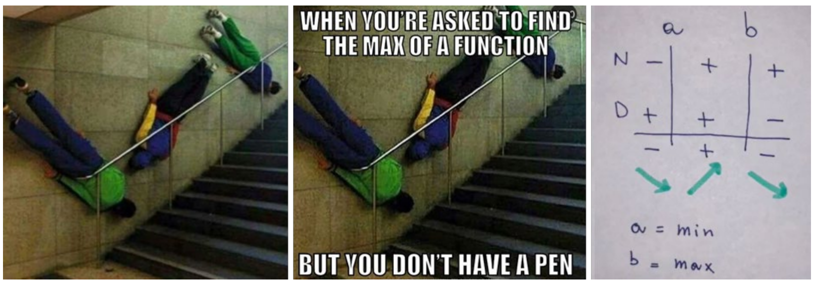

Figure 4: Mario's template (left) finished meme (centre) and video explanation (right)

Mario's attention is seized by the template (Figure 4, left), which he *sees as* (his own words!) the graphic representation of a function. He neatly captures the role of the culture in interpreting the image, picturing that *a normal person* would not see in it more than people *in strange positions*, while a mathematically infused observer (*a math student*) is reminded of the graph of a function with a maximum and a minimum. However, his understanding appears confused: he shifts from referring to the cartesian graph of the function itself to referring to the derivative sign chart needed *to find the maximum and minimum of a function,* which is the idea apparently addressed by the meme (Figure 4, centre) and developed in the explanatory video (Figure 4, right). We can hypothesise that Mario's confusion is induced by the similar appearance between the possible cartesian graph of a function *following the lines of the bodies* in the template (such as $y = x - x^3$) and the arrows in its derivative sign chart. Arrows in his video match the bodies in the template but show another misconception: they describe the increasing and decreasing trend of a quotient (N and D stand for numerator and denominator, and the chart corresponds to the sign rule for division), but the value b where the denominator of the derivative changes its sign is not necessarily a maximum for the function. This analysis reveals that Mario's mathematical prior is deeply influenced by the iconic aspect of the mathematical concept he wanted to represent, a fact that hinders his understanding and produces the confusion.

Luca's mathematical meme

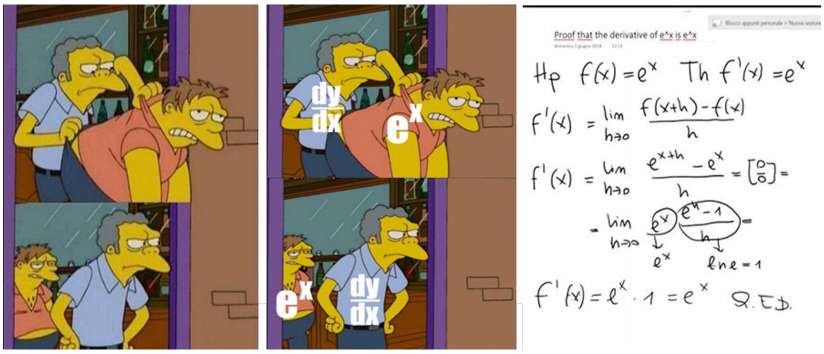

Figure 5: Luca's template (left) finished meme (centre) and video explanation (right)

Luca's meme (Figure 5, centre) addresses the fact the derivative of the exponential function $y = e^x$ is the function itself. Luca's template (Figure 5, left) is a two-panel image to be read from top to bottom: in the upper frame a bartender is throwing a customer out the bar, and in the lower frame the customer reappears undisturbed behind the bartender. This template is habitually used to represent recurring events: in this case, the mathematical fact that $\frac{d(e^x)}{dx} = e^x$, which is subsequently proved by Luca in his explanatory video (Figure 5, right).

> Researcher: The same question for you: how did you get your idea for the meme?
>
> Luca: I did not have any idea yet... so I was looking among the various templates to see if any could particularly inspire me and then I saw this template [Figure 5, left], which by the way I had actually already thought of, that is, I had already seen that template and something about it had already occurred to me before doing this work on memes... so as soon as I found the template it was an epiphany... because after having worked on it so much it was automatic, that is, the image itself unleashed…
>
> Researcher: So, it's the same thing ...
>
> Luca: Yes, it is the image that brings [the mathematical concept] to your mind... […] it is precisely the image and the mathematical topic that are connected, almost naturally

Similar to Mario, Luca's attention is captivated by a template (Figure 5, left) discovered while scrolling the Internet, activating a mathematical idea. Luca acknowledges the role of the culture in his vision (*after having worked on it so much it was automatic)* but it's important to highlight that Luca's *epiphany* is sparked by the metaphorical value of the image, representing recursion, rather than by its iconic value, as is the case with

Mario's meme. This reveals that Luca's mathematical prior involves a conceptual understanding of the mathematical idea represented by the meme, that attains to the deep mathematical structure and not simply to its surface appearance. This conceptual understanding is confirmed in the video explanation where he faultlessly proves that the "derivative of e^x is e^x" (Figure 5, right).

RESULTS AND DISCUSSION

The analysis of the mathematical memes created by Mario and Luca provides valuable insights into how students' mathematical prior influences the way they see templates as representations of mathematical concepts. Comparing Mario's and Luca's cases, it becomes evident that students' mathematical priors, constituted of conceptual understanding of the topic, cultural context, and educational experiences, play a crucial role in shaping their interpretation of visual stimuli. Borrowing Etkind and Shafrir's terminology (2013, p. 5347), we can affirm that Luca is a *good conceptual thinker* and his mathematical prior allows him to recognise the *meaning equivalence* between the template and the mathematical topic, while Mario is a *poor conceptual thinker* and is misled by the *surface similarity* between the template and contrasting mathematical concepts. Acknowledging the limitation of the sample cases, these examples illustrate the significance of analysing how students interact with visually ambiguous stimuli, such as meme templates, to gain insights into their knowledge and understanding of mathematical concepts, which is something that is not so straightforward to achieve with traditional tasks. The analysis also uncovers students' strengths and weaknesses, such as misconceptions and cultural influences on students' understanding of mathematical concepts. The findings from this research have implications for educators in planning interventions to address misconceptions and select further attuned visual inputs to support students' meaning-making.

In conclusion, the analysis of meme templates as visually ambiguous stimuli offers a unique lens through which to explore students' understanding of mathematical concepts. The insights gained from this study can contribute to the broader conversation on the role of visual inputs in mathematics education and provide practical considerations for educators seeking to enhance students' mathematical understanding**.**

References

Andrá, C., Lindström, P., Arzarello, F., Holmqvist, K., Robutti, O., & Sabena, C. (2015). Reading mathematics representations: An eye-tracking study. *International Journal of Science and Mathematics Education, 13*(2), 237–259.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educ. Stud. Math, 52*, 215–241. https://doi.org/10.1023/A:1024312321077

Arzarello, F., Robutti, O., & Bazzini, L. (2005). Acting is learning: focus on the construction of mathematical concepts. *Cambridge Journal of Education*, *35*(1), 55-67.

Bernstein, D. A. (2008). *Essentials of Psychology*, 123–124. Houghton Mifflin Company https://sangu.ge/images/EssentialsofPsychology.pdf

Bini, G., Robutti, O., & Bikner-Ahsbahs, A. (2022) Maths in the time of social media: conceptualizing the Internet phenomenon of mathematical memes. *Int J Math Educ Sci Technol, 53*(6), 1257–1296. https://doi.org/10.1080/0020739X.2020.1807069

Bini, G., Bikner-Ahsbahs, A. & Robutti, O. (2023). "How to meme it": reverse engineering the creative process of mathematical Internet memes. *Educational Studies in Mathematics, 112*(1), 141–174. https://doi.org/10.1007/s10649-022-10173-1

Brugger, P., & Brugger, S. (1993). The Easter bunny in October: Is it disguised as a duck? *Perceptual and motor skills*, *76*(2), 577–578.

Cecchinato, G. (2009). Tags: media, conoscenza [Tags: media, knowledge]. *Bibliotime, XII*(2). https://www.aib.it/aib/sezioni/emr/bibtime/num-xii-2/cecchina.htm

Cohen, L., Lawrence, M., & Keith, M. (2007). *Research Methods in Education. 6th Edition*. Taylor and Francis Group.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st North American PME Conference, 1*, 3–26.

Esposito, A., Chiarella, S. G., Raffone, A., Nikolaev, A. R., & van Leeuwen, C. (2023). Perceptual bias contextualized in visually ambiguous stimuli. *Cognition, 230*, 105284. https://doi.org/10.1016/j.cognition.2022.105284

Etkind, M., & Shafrir, U. (2013). Teaching and learning in the digital age with pedagogy for conceptual thinking and peer cooperation. In INTED2013 Proceedings (pp. 5342–5352). https://library.iated.org/view/ETKIND2013TEA

Felten, P. (2008). Visual literacy, change: *The Magazine of Higher Learning, 40*(6), 60–64. https://doi.org/10.3200/CHNG.40.6.60-64

Jastrow, J. (1899). The Mind's Eye. *Popular Science Monthly, 54,* 299–312.

Kihlstrom, J. F. (n.d.). *Duck-rabbit illusion*. Retrieved from https://www.ocf.berkeley.edu/~jfkihlstrom/JastrowDuck.htm

Mitchell, W. J. T. (1994). *Picture theory: Essays on verbal and visual representation*. University of Chicago Press. https://press.uchicago.edu/ucp/books/book/chicago/P/bo3683962.html

Presmeg, N. (2006). Research on Visualization in Learning and Teaching Mathematics. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205–236). Sense.

Radford, L. (2010). The eye as a theoretician: seeing structures in generalizing activities. *For the Learning of Mathematics, 30*(2), 2–7.

Seriès, P., & Seitz, A. R. (2013). Learning what to expect (in visual perception). *Frontiers in human neuroscience*, *7*, 668. https://doi.org/10.3389/fnhum.2013.00668

Wittgenstein, L. (1953). *Philosophical Investigations: The German Text, with a Revised English Translation.* Blackwell.

CONTEMPLATING THE ROLE OF MATHEMATICAL EGOTISM

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This theoretical paper considers Mathematical Egotism's role in the development and reinforcement of particular views of mathematics, beliefs about mathematics, and students' beliefs about themselves. Various literature, examples, and considerations are presented in exploring how Mathematical Egotism contributes to students' disengagement, disillusionment, or disaffection with mathematics. This paper also provides a possible antidote via Mathematical Empathy.

INTRODUCTION

The mention of mathematics has the potential to stir a range of responses. Some speak of mathematics' beauty, while others speak of its dull irrelevance and elitism (Nardi & Steward, 2003). Beliefs – or cognitive filters that consciously or unconsciously influence interpretations of new phenomena (Calleja, 2021) – can be gained from life experiences and shape a person's identity or perceived place in the world (Collinson, 2012). Mathematics classrooms are one place where students may develop their beliefs regarding mathematics and themselves. While classrooms are intended to be a place of support, the underlying existence of Mathematical Egotism can unintentionally create barriers that lead to students' disengagement, disillusionment, and disaffection towards mathematics. The purpose of this theoretical paper is not to shame or blame busy teachers but to consider the possibility of Mathematical Egotism's role in developing and reinforcing particular views of mathematics, beliefs about mathematics, and students' beliefs about themselves. For this reason, this paper will first position the author and then investigate the terms ego and egotism, before identifying the presence of Mathematical Egotism through the provision of examples. The paper will conclude by offering a possible antidote in Mathematical Empathy.

POSITIONING MYSELF

Following Thanhesier's (2023) lead in sharing her position before discussing related topics, my experiences and the development of beliefs will also be acknowledged with respect to a student of mathematics, a teacher of mathematics, and a researcher of mathematics education. I did not experience much success in school mathematics, but not for the lack of trying. Mathematics did not make sense. The discussions were brief, and the textbook exercises were never-ending. I only found out I was practising the wrong approaches to mathematical problems after test results or assignment grades were released, with no time to ask for help because the next unit was beginning. Nevertheless, I kept trying because I wanted to be a person who understood mathematics and intended to be a teacher. After failing my first assessment piece in Term 1 of Senior studies, I was encouraged to drop from a higher track in mathematics

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 104–111). PME.

to an “easier” track. This was the last straw for my confidence, and soon, I began treading along the failing line of this lowest track, too.

The disengagement and disaffection I experienced in school mathematics are not unique. Mathematics is experienced and perceived as a boring, difficult subject with little relevance and value (Nardi & Steward, 2003). Mathematics can be viewed as quite a conservative subject shrouded in “tradition” or universal truths (Handal, 2009) and commonly perceived as the rote-learned reproduction of pre-established procedures and rules (Calleja, 2021; Nardi & Steward, 2003; Shepard, 2000). Mendick (2005) alleges that this view of mathematical knowledge “as absolute and unquestionable” creates the status of “the ultimate intelligence test” (p. 247). Consequently, there are beliefs that mathematics is reserved for “maths people” (Palmer, 2009), with “maths brains” (Boaler, 2015), and the continuation of elitism, superiority and stereotypes (Nardi & Steward, 2003) reiterated with each generation.

As a student of mathematics, I was often surrounded by teachers who told me how much they loved mathematics. I wondered how they could ever understand my perspective when we seemed to have very different experiences of mathematics. Then - later on - as a teacher, I found myself speaking with students who shared similar and familiar wonderings, some of whom assumed I must have had undisputed success in mathematics class to eventually teach mathematics. While my career and further studies took me to different places, I now find myself undertaking a PhD in mathematics education to spend my career in mathematics education research. However, there is a wariness of working amongst a field of researchers who – upon my school mathematics experiences and the societal messages I am predisposed to – I may not authentically belong with. I find myself straddling multiple standpoints - the mathematics education researcher, the teacher, and the students who are disengaged, disillusioned, or disaffected with mathematics. While my experiences as a mathematics education researcher have been unanticipatedly inclusive, the beliefs and values constructed in school mathematics continue to linger and compete with what I now know.

My PhD research aims to interrogate the meaning of success in mathematics education. Though broadly cited, there are contradictory and conflicting messages about various success phenomena in mathematics education or how success is determined—for example, success metrics or assessment. However, the recurrence and variety of applications suggest that success is highly sought after and highly valued. Mathematics education research has increasingly acknowledged the influence of values and beliefs in mathematics education, including the interconnectedness of affective domains, beliefs around the nature of mathematics and the different theoretical or practical approaches to mathematics (Beswick, 2021; Bishop et al., 2003). Despite the close relationship learning, pedagogy, and assessment should share, acknowledging the influences of values and beliefs within assessment is not nearly as commonly appraised (Musial, 2021). As the PhD exploration continues, several peripheral observations have emerged and provoked questioning of taken-for-granted psychological ideologies intertwined with success, metrics of success and the assessment of success. For example, questioning whether an unwillingness to let go of behaviourist assessment ideologies constrains the advancement of school mathematics education, further contributing to the narrowing of beliefs about success (Burtenshaw, 2023). This theoretical paper would like to consider another peripheral observation by exploring the role of Mathematical Egotism.

EGO AND EGOTISM

Ego and Egotism share similar derivation, though they have distinct expositions theoretically. Freud (1923) establishes three enacting aspects within a personality structure model - the id, the ego and the superego. The id is the most unconscious and primal part of our psyche, often associated with unbridled pleasure and desires (Freud, 1923). The ego is a “part of the id which has been modified by the direct influence of the external world” (Freud, 1923, p. 25). Broadly, the ego assists in negotiating, counteracting or - at times - resisting the primal or impulsive desires of the id and acts as the “processing system” or decision-making component that strives to fulfil the id in the most socially appropriate manner (Freud, 1923). A strong ego is good, as a strong ego is more apt at rationalising primitive desires. The Superego embodies and absorbs society's broader values, morality, ideals and ethics, which provides standards for the ego (Allison, 2023). The voice of consciousness develops from a young age as our psyche assimilates and conforms to form an ideal sense of self (Allison, 2023; Freud, 1923). Freud’s seminal theories continue to be challenged and re-examined (e.g. Allison, 2023). However, like many others, Freud offers a foundation for further exploration. These explorations of ego’s origins provide an alternative perspective to the commonly used applications of ego, whereby the term egotism is more fitting.

Broadly, egotism is driven by a desire for “social status, glory, credit, adulating attention, honour, superiority, special entitlements, prestige, and power” (Roberts & Cleveland, 2017). Egotism often manifests with accompanying or contributing vices such as arrogance, haughtiness, defensiveness, vanity or narcissism - the latter driven explicitly by superiority or a desire to be “accepted as a member of one’s affinity groups” (Tanesini, 2021, p. 97). Through exploring types of egotism comprised of vices of superiority (see Tanesini, 2021), Coppola (2023) writes how “a person confirms their self-beliefs of superiority by engaging in downward social comparison” (p. 299). As such, egotism increases the potential of enacting humiliation and intimidation tactics, leading to potential alienation of self or others (Coppola, 2023). Similar behavioural traits can be applied explicitly to and observed within Mathematical Egotism.

MATHEMATICAL EGOTISM

Mathematical Egotism is the tendency to depict mathematics - often from an exclusive perspective - in a manner that alienates different perspectives, beliefs or values and which contributes to the sustained elitism and superiority for which mathematics is societally notorious. Mathematical Egotism does not refer to the person or people but a trait incidentally adopted by a person. As the discussion on egotism suggests, Mathematical Egotism could leverage mathematics to secure or demonstrate social status, distinction, superiority and prestige. Mathematical Egotism can be overt or very subtle and could be unintended or - to a lesser extent - intended. Therefore, Mathematical Egotism can manifest in diverse ways, including through a continued reputation of elitism, a lack of self-awareness (of self and others), and alienating pedagogical practices.

Social status, superiority and prestige within Mathematical Egotism are not modern phenomena. Burnyeat (2000), in his paper Plato on Why Mathematics is Good for the Soul, describes how Plato believed that a person could not become “a moral hero or saint” without “discipline in sheer hard thinking”, such as what mathematics provides. He writes on the various opinions of similarly prestigious minds of Ancient Greece, all of which speak of sharp minds, natural intelligence and how mathematics studies “is extremely demanding to learn and practice” and, therefore, is “a good test of intellectual and moral calibre” (Burnyeat, 2000, p. 9). Moral righteousness and strong mathematical capabilities are interwoven.

An example of Mathematical Egotism - though small but repetitively observed - is the frequency with which confident mathematicians or mathematics teachers discuss their appreciation for mathematics while also - possibly unwittingly - alienating people or their students for not similarly beholding such fondness. To be clear, sharing one’s passions is not to be reprimanded, nor is celebrating passions inadmissible. But, for the sake of discussion, let us consider possible assumptions within commonly occurring statements such as “Why can’t my students see the beauty and creativity of mathematics?” or “Mathematics is fun and engaging on its own and does not require real-world applications” (see Montano, 2013). Firstly, this denotes that there is a universal understanding of what mathematics is. Thanheiser (2023) writes how “the field of mathematics is in agreement that there is no joint definition of what mathematics is” (p. 1). Teachers’ beliefs about the nature of mathematics and what they value regarding mathematics will guide how teachers teach the subject (Beswick, 2012; Bishop et al., 2003; Calleja, 2021). Literature (e.g. Calleja, 2021; Ernest, 1989; Thanheiser, 2023) also depicts varying approaches to the learning and teaching of mathematics, often each with their own beliefs and values underpinning (Shepard, 2000). This begs the question, if there is more than one view of mathematics and more than one way to learn or teach mathematics, could there be more than one way of becoming a self-assured mathematician?

These passing statements about mathematics also assume that the stated views of mathematics and beliefs align with what is demonstrated through teachers’ pedagogical decision-making, assessment practices and classroom culture. Calleja (2021) and Beswick (2012) have noted similar unalignment between teacher beliefs and practices in their research. Another assumption is that statements like these propose that students who are disengaged, disillusioned, or disaffected with mathematics choose not to have a positive relationship with mathematics or do not want to see the applicability of mathematics. In this aspect, students shoulder the responsibility to make sense of mixed messages while managing disillusionment, along with the blame when they are unable to reconcile unaligned experiences. This seems counterintuitive. Not recognising these possibilities or assumptions could unintentionally lend itself to Mathematical Egotism.

Equivalent examples may include a wealthy business executive telling a destitute person how easy it is to make a million dollars. Alternatively, a very fit person flexes and boasts to a friend who openly struggles with their fitness goals or body image. As with Mathematical Egotism, circumstances or barriers may exist that are not considered in each circumstance. For example, the one (presumptively in a "superior" position) has a deeper understanding of critical aspects the other has not yet grasped or different experiences that have led to the development of beliefs.

Mathematical Egotism could be operating within classroom culture and relationships with mathematics educators. Mathematical Egotism may emerge when, after a student demonstrates how they went about a mathematical task to the class, their whiteboard work is promptly erased and replaced by the teacher showing "the best way" or "the quickest way" of reaching that same solution. Again, while rational and possibly justifiable, there is deliberation to consider how Mathematical Egotism contributes to the developing beliefs of students and their relationship with mathematics. Coincidentally, Gatekeeping was also a theme that arose in Coppola's (2023) exploration of musical egotism and how "one's expression of superiority might be interpreted as an implicit degradation of others' self-worth" (Leary et al., 1997 in Coppola, 2023). She found that participants experienced egotism through poor relationships with teachers and a poor sense of belonging in the classroom, which was "very often perpetuated through hierarchical master-apprentice relationships with teachers" (p. 306). Finally, Coppola (2023) also discusses the interwovenness of egotism and elitism. The latter has already been determined via research as contributing reasons for the disaffection towards mathematics (Nardi & Steward, 2003).

Mathematical Egotism may be experienced when competent mathematicians –students or professional adults - work on a group task amongst a mixed-ability group yet speed ahead and disregard the peers left behind. This is not to say capable mathematicians should hide their expertise, but motivations or intentions and whether these meet the goals of the group task should be considered. This example does not intend to encourage notions of streaming. However, streaming culture in schools could also have traits of Mathematical Egotism via the perceived rank and stereotypes associated with this culture.

There is theoretical and experiential evidence to suggest Mathematical Egotism could be a blind spot within our wider profession. An awareness of Mathematical Egotism could create a space to explore the barriers for students who are disengaged, disillusioned, and disaffected with mathematics. However, this exploration could also determine that Mathematical Egotism is a contributing barrier for students who are disengaged, disillusioned, and disaffected with mathematics. For example, when a student is vulnerable and shares their disaffection, how do we, as a profession, regard those beliefs, or are these disclosures dispelled as they differ from the teachers' beliefs and experiences? How often do we consider poor behaviour as, alternatively, the "quiet, invisible disaffection" that Nardi and Steward (2003) describe? Do students perceive the open invitation to admire mathematics as authentic, or do such invitations lead to further alienation? The antidote to such Mathematical Egotism may lie within Mathematical Empathy.

MATHEMATICAL EMPATHY

Confident mathematicians and mathematics teachers are in a powerful position to support students who are disengaged, disillusioned, or disaffected with mathematics. Their depth of mathematical knowledge, love for mathematics, and associated skill sets like problem-solving signify their unique perspicacity to support students through their disillusionment and mathematics past this negative reputation. An adjustment from ego-centredness to other-centredness (Davis, 2010) could begin to write a new narrative - from Mathematical Egotism to Mathematical Empathy.

Empathy is the ability to vicariously recognise and acknowledge the feelings and experiences of others, including the cognitive capacity to infer thoughts and beliefs (Williams, 2019). Commonly depicted as "the ability to walk in another's shoes", empathy is notably different to sympathy, which typically denotes feelings of pity for another's misfortune and - in some cases - relief that one is not in the "other's shoes". Empathy takes an approach grounded in equality or equity, whereas sympathy has an implicit dynamic imbalance. Stojiljković et al. (2012) explain how empathy is an important characteristic of effective teachers and opens communication between teachers and students.

Mathematical Empathy refers to the mindfulness of others' mathematical perspectives, beliefs and experiences with mathematics. There is difficulty in distinguishing the path ahead when you are lost. However, mathematicians and mathematics teachers have a greater understanding of mathematics and the paths towards that greater understanding. Instead of mathematicians and mathematics teachers imposing their own views of mathematics and beliefs about mathematics - which Mathematical Egotism traits could drive - Mathematical Empathy sees the teacher walk alongside the student with a shared goal of facing or overcoming barriers that lead to disengagement, disillusionment or disaffection.

CONCLUSION

The phenomenon of Mathematical Egotism may be confronting or uncomfortable to contemplate. This paper is not to condemn but to consider. In considering Mathematical Egotism, this paper discussed how its existence could result in the tendency to depict mathematics in a manner that alienates different perspectives of beliefs, contributing to the sustained elitism and superiority for which mathematics is societally notorious. Teachers' awareness of their own beliefs relating to mathematics and those of their students could help teachers recognise instances of unintentional Mathematical Egotism and its possible contribution to students' disengagement, disillusionment, and disaffection with mathematics. This awareness forms the foundations of Mathematical Empathy – providing an antidote to Mathematical Egotism – as teachers work alongside students to disentangle their views, beliefs, and self-perceptions pertaining to mathematics. Mindfulness of Mathematical Egotism could provide a fundamental step towards shifting narratives and broadening the opportunities for success within mathematics education.

References

Allison, E. (2023). "The Ego and the Id": How and why Freud transformed his model of the mind. *The International Journal of Psychoanalysis, 104*(6), 1063–1076.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127–147.

Bishop, A., Seah, W. T., & Chin, C. (2003). Values in mathematics teaching—The hidden persuaders? *Second International Handbook of Mathematics Education*, pp. 717–765.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching.* John Wiley & Sons.

Burnyeat, M. (2000). Plato on why mathematics is good for the soul. In T. Smiley (ed.), *Mathematics and Necessity: Essays in the History of Philosophy*. pp. 1-81

Burtenshaw, R. (2023). The Deeply Engrained Behaviourist Assessment Ideologies Constraining School Mathematics. In B. Reid-O'Connor, E. Prieto-Rodriguez, K. Holmes, & A. Hughes (Eds.), *Weaving mathematics education research from all perspectives. Proceedings of the 45th annual conference of the Mathematics Education Research Group of Australasia* (pp. 123–130). Newcastle: MERGA.

Calleja, J. (2021). Changes in mathematics teachers' self-reported beliefs and practices over the course of a blended continuing professional development programme. *Mathematics Education Research Journal*, pp. 1–27.

Collinson, V. (2012). Sources of teachers' values and attitudes. *Teacher development, 16(*3), 321–344.

Coppola, W. J. (2023). The essence of arrogance: A phenomenology of musical egotism. *Research Studies in Music Education*, 45(2), 298-314.

Davis, H. (2010). Other-centredness as a leadership attribute: From ego to eco centricity. *Journal of Spirituality, Leadership and Management, 4*(1), 43–52.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–253). Falmer.

Freud, S. (1923). The Ego and the Id. *The Standard Edition of the Complete Psychological Works of Sigmund Freud, Volume XIX* (1923-1925*): The Ego and the Id and Other Works*, pp. 1-66

Handal, B. (2009). Philosophies and pedagogies of mathematics. *Ilköğretim Online*, *8*(1), 1–6.

Montano, U. (2013). *Explaining beauty in mathematics: An aesthetic theory of mathematics* (Vol. 370). Springer Science & Business Media.

Meisner, G. B. (2018). *The golden ratio: The divine beauty of mathematics*. Race Point Publishing.

Mendick, H. (2005). Mathematical stories: Why do more boys than girls choose to study mathematics at AS-level in England?. *British Journal of Sociology of Education*, *26*(2), 235–251.

Musial, D. L. (2021). Meaningful assessment. In L. Wellner & K. Pierce-Friedman (Eds.), *Supporting early career teachers with research-based practices* (pp. 50-70). IGI Global.

Nardi, E., & Steward, S. (2003). Is mathematics TIRED? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal,* 29(3), 345–367.

Palmer, A. (2009). 'I'm not a "maths-person"!' Reconstituting mathematical subjectivities in aesthetic teaching practices. *Gender and Education, 21*(4), 387–404, DOI: 10.1080/09540250802467950

Roberts, R. C., & Cleveland, W. S. (2017). Humility from a philosophical point of view. In E. L. Worthington, Jr., D. E. Davis, & J. N. Hook (Eds.), *Handbook of humility: Theory, research, and applications* (pp. 33–46). Routledge.

Shepard, L. A. (2000). The role of assessment in a learning culture. *Educational Researcher*, *29*(7), 4–14. https://doi.org/10.3102/0013189X029007004

Stojiljković, S., Djigić, G., & Zlatković, B. (2012). Empathy and teachers' roles. *Procedia-social and behavioral sciences*, 69, 960-966.

Tanesini, A. (2021). *The mismeasure of the self: A study in vice epistemology*. Oxford University Press.

Thanheiser, E. (2023). What is the Mathematics in Mathematics Education?. *The Journal of Mathematical Behavior*, 70, 101033.

Wilkie, K. J., & Tan, H. (2019). Exploring mathematics teacher leaders' attributions and actions in influencing senior secondary students' mathematics subject enrolments. *Mathematics Education Research Journal*, 31(4), 441-464.

UNDERSTANDING STUDENTS' REASONS AND AIMS OF EFFORTS AND PERSISTENCE IN MATHEMATICS

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In the present study, we investigate the 'aims' elementary school students pursue through effort and persistence (i.e., their achievement goals) and the 'reasons' driving them (i.e., their motivations) in their mathematics classes. Self-report instruments measuring students' motivational reasons, achievement goals, and effort and persistence in their mathematics classes were administered. Mediational path analysis showed that achievement goals, collectively, played a significant mediating role in almost all the links connecting motivational reasons to effort and persistence. Autonomous motivation was associated with greater effort and persistence. Self-based goals strengthened the positive direct effects of autonomous motivation on effort and persistence.

INTRODUCTION

Singapore students have consistently performed well in the international assessment of academic proficiencies (see e.g., results of the 2019 Trends of International Mathematics and Science Study [TIMSS]; Mullis et al., 2020). However, there is a possibility that students find that their parents and teachers set high academic expectations which are communicated in a controlling way. Furthermore, students perceive teacher-led whole class instruction during mathematics classes as a norm (Kaur & Ghani, 2012) and teaching approaches tend to be highly structured. Bearing in mind these characteristics of Singapore's sociocultural and educational system, this study aims to explore Singapore's students' effort and persistence in mathematics in relation to their achievement goals and motivational reasons. Findings of this study will lend practical insights into fostering engagement of students with similar cultural and educational characteristics in their learning of mathematics.

Underpinned by self-determination and achievement goal perspectives (Elliot, Murayama, & Pekrun, 2011; Ryan & Deci, 2000), we investigated the 'what' and the 'why' of students' effort and persistence by studying the 'aims' students seek to pursue through effort and persistence (i.e., their achievement goals) and the 'reasons' driving their effort and persistence (i.e., their motivation). As shown in Figure 1, the role of achievement goals in mediating the effects of motivational reasons on effort and persistence in mathematics is examined.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 112–119). PME.

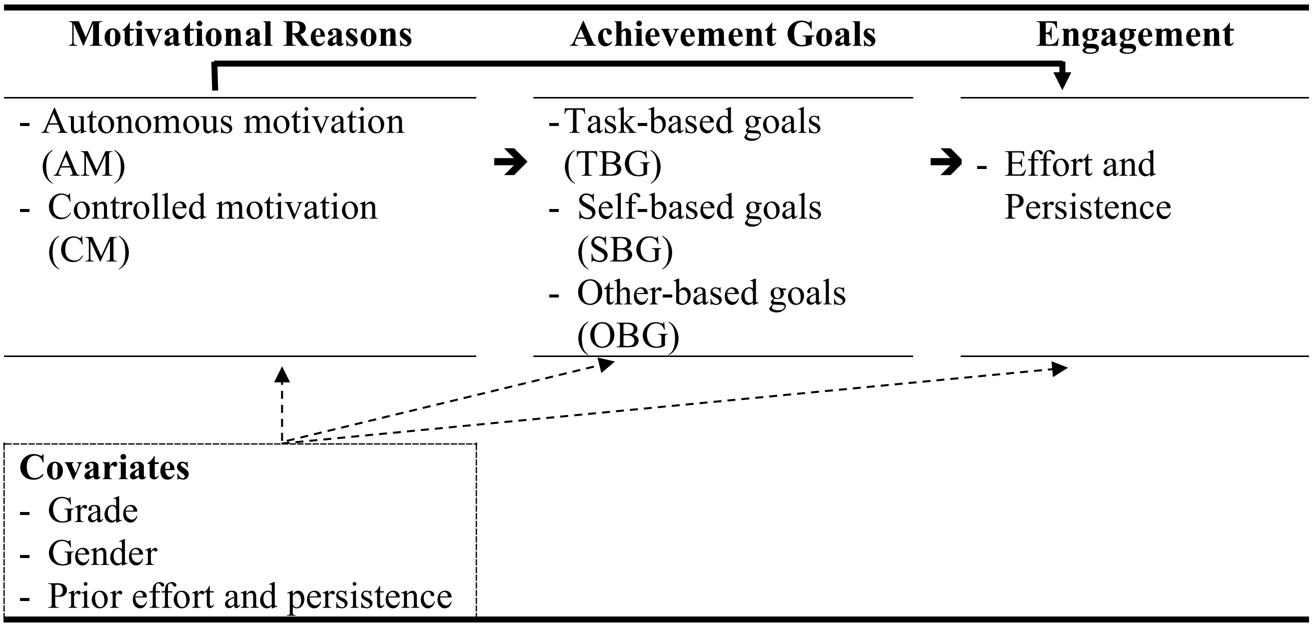

Figure 1: Hypothesised model depicting the role of achievement goals in mediating the links between motivational reasons and effort and persistence (controlling for sociodemographic factors and prior effort and persistence)

THEORETICAL FRAMEWORK

Achievement Goals

In the 3 × 2 achievement goal model, Elliot et al. (2011) proposed six types of achievement goals which students adopt. They are task-approach goals which orient students to attain the task's absolute demands; task-avoidance goals which orient students to avoid not attaining the task's absolute demands; self-approach goals which orient students to score better than their own previous score; self-avoidance-goals which orient students to avoid scoring worse than previous score; other-approach goals which orient students to score better than others their peers; and other-avoidance goals which orient students to avoid scoring worse than their peers. A study conducted by Johnson and Kessler (2013) found that task-based goals had a more positive relationship with academic achievement than self-based goals whereas other-based goals had a negative relationship with academic achievement.

Self-determined motivation as reasons underlying achievement goals

Several researchers (e.g. Vansteenkiste et al., 2010) found that different motivations underlying achievement goals differentially predicted educational outcomes. Self-determination theory (Ryan & Deci, 2000) maintains that the reasons that regulate or motivate individuals' behaviours may vary in their level of autonomy. Autonomous motivation refers to a motivational source that regulates individuals' behaviours because of the enjoyment and fulfilment of personal meaning that the activity or behaviour serves to bring. Meanwhile, controlled motivation represents a motivational source that regulates individuals' behaviour because of the intention to meet perceived social expectations (such that enticing the behaviour brings pride to themselves rather than guilt or humiliation) and to obtain rewards and/ or avoid punishment. Research has shown that relative to controlled motivation, autonomous motivation was associated with stronger intention to persevere in studies (Lavigne et al., 2007).

Autonomous and controlled motivations have recently been viewed as possible motivational reasons underlying achievement goals (Vansteenkiste et al., 2014) and they are aligned with the hierarchical model of achievement motivation (Elliot, 1999) where students' achievement goals mediate the relationships between their individual attributes (e.g., achievement motive) and engagement. Students' engagement could be understood from the reasons driving engagement and the aims they seek to attain through engagement. For instance, students striving to outperform their peers (i.e., pursuing other-approach goals) may do so because they do not wish to lose extra play time (i.e., controlled reason) or the personal benefits they anticipate for furthering their education (i.e., autonomous reason). Students seeking to understand learning materials (i.e., pursuing task-approach goals) may do so out of their intention to meet a deadline (i.e., controlled reason) or their interest in the topic (i.e., autonomous reason).

Prior studies have sought to integrate key constructs in the self-determination and achievement goal perspectives. Vansteenkiste et al. (2010) found, beyond Belgian students' performance-goals, autonomous motivation underlying performance-approach goals predicted persistence. Michou et al. (2014) found that for Greek university students, controlling reasons for mastery-approach goals were predicted by fear of failure and related negatively to effort regulation. The same study also found that controlled motivation negatively predicted effort regulation. Cai and Liem (2017) found that self-based goals strengthened the benefits of autonomous motivation on elaboration among elementary mathematics students. Taken together, autonomous motivation underlying achievement goals are more beneficial for students' learning than controlled motivation. As such, we ask: In the Asian context, which achievement goal plays a more important mediating role in the link between motivations and effort and persistence among elementary school students' learning of mathematics?

METHODOLOGY

Participants and Measures

The sample comprised 491 students (54% girls; $M_{\text{age}} = 11$, $SD_{\text{age}} = 0.87$) from a Singapore government elementary school selected based on convenience sampling. Around 35% of the participants were in Grade 4, 38% in Grade 5, and 27% in Grade 6.

The Academic Self-Regulation Questionnaire (Ryan & Connell, 1989) measuring motivational reasons, 3 × 2 Achievement Goal Questionnaire (Elliot et al., 2011) and Student Approaches to Learning (Marsh et al., 2006) subscale for effort and persistence were adapted to the mathematics classroom context and administered to the students. Earlier studies which have used these scales have documented their validity and reliability. The items for motivational reasons and achievement goals were rated on a scale ranging from 1 (not at all true of me) to 5 (very true of me), while effort and persistence items were rated on a scale ranging between 1 (never) and 5 (always). In the Academic Self-Regulation Questionnaire (Ryan & Connell, 1989), a 17-item questionnaire, autonomous motivation was operationalised by six intrinsic and identified motivation items (α = 0.84), while controlled motivation was measured by the 11 introjected and extrinsic motivation items (α = 0.85). The 3 × 2 Achievement Goal Questionnaire (Elliot et al., 2011) consists of 18 items incorporated into six three-item goal subscales: Task-approach, task-avoidance, self-approach, self-avoidance, other-approach, and other-avoidance goals. The three six-item higher order achievement goal subscales also displayed sound reliability, task-based goals (α = 0.82), self-based goals (α = 0.88), and other-based goals (α = 0.89). Effort and persistence were measured by the four-item effort and persistence subscale (α_{T1} = 0.83, α_{T2} = 0.83) obtained from the Student Approaches to Learning instrument (Marsh et al., 2006).

Procedure

The students completed the survey in the second and third school terms of the year (i.e., Time-1 and Time-2 data, respectively). Although this study focused on the third school term data, the research design allowed us to control for effort and persistence outcome factors measured at the second school term. The significant effects on motivation and achievement goals on effort and persistence would then be considered practically important as they are 'over and above' the effect of each corresponding prior effort and persistence factor.

RESULTS AND INTERPRETATION

As presented in Table 1, the approach and avoidance dimensions of an achievement goal type were found to be highly correlated (r = 0.82 between task-approach and task-avoidance goals; r = 0.84 between other-approach and other-avoidance goals; r = 0.97 between self-approach and self-avoidance goals), showing that elementary school children in this sample were unable to distinguish between the approach and avoidance dimensions of each achievement goal type. Therefore, the main analysis focused on three distinct types of achievement goals.

Path analysis testing the multiple mediation of the three types of achievement goals in linking motivation to effort and persistence was performed using the bootstrapping method, a non-parametric procedure that does not require the assumption of normality of sampling distribution. Using the Mplus syntax for multiple mediation provided by Preacher and Hayes (2008), this approach would enable us to estimate parameters of both total indirect effects associated with all the three achievement goals and specific indirect effects via each achievement goal. Parameter estimates, and 95% bias-corrected confidence intervals of the indirect effects were generated from 5000 bootstraps (random samples). Mediation occurs when indirect effects are significant. Effort and persistence were estimated while controlling for prior (Time-1) effort and persistence and covariates (grade, gender).

Table 1 shows the parameter estimates in the mediational models. Autonomous and controlled motivations significantly predicted achievement goals. Beyond the effects of the covariates and prior effort and persistence, self-based goals were the only type of achievement goal that significantly predicted effort and persistence (β = 0.22).

	Covariates		
	Grade	Gender	Prior effort and persistence
Predicting effort and persistence			
AM	-0.20***	0.01	0.42***
CM	-0.27***	-0.01	0.21***
TBG	0.06	0.10*	0.12**
SBG	0.07*	0.15**	0.15***
OBG	-0.01	-0.03	0.05
Effort and persistence	0.02	0.02	0.34***

	Motivational reasons		Achievement goals		
	AM	CM	TBG	SBG	OBG
Predicting effort and persistence					
AM					
CM					
TBG	0.21***	0.21***			
SBG	0.20***	0.12***			
OBG	0.10*	0.40***			
Effort and persistence	0.32***	0.03	0.08	0.22***	0.05

Note: *p < 0.05, **p < 0.01, p*** < 0.001; Gender (1 = boy, 2 = girl); prior effort/ persistence refers to the effort/ persistence measured three months earlier.

Table 1: Summary of standardised regression coefficients (betas) in the path models predicting students' effort and persistence.

Table 2 presents the direct, indirect, and total effects in the mediational models. The total indirect effects of the two motivational reasons on effort and persistence were significant in both instances, reinforcing the importance of achievement goals, collectively, in linking motivation to effort and persistence. However, self-based goals were the only achievement goal type which significantly mediated these relationships.

The total effect of autonomous motivation on effort and persistence was positive (β = 0.39), resulting from its direct effect (β = 0.32) and indirect effect through self-based goals (β = 0.07). Although the total and direct effects of controlled motivation on effort and persistence were not significant, its indirect effect through self-based goals was significantly positive (β = 0.03).

	Direct effect	Indirect effect through				Total effect
		TBG	SBG	OBG	All AG	
Predicting effort and persistence						
AM	0.32***	0.02	0.04**	0.01	0.07***	0.39***
CM	-0.03	0.02	0.03**	0.02	0.06***	0.03

Table 2: Summary of effects in the mediation path models predicting effort and persistence.

Discussion and Applied Implications

Findings show that after controlling for sociodemographic and prior effort and persistence, autonomous motivation is associated with heightened effort and persistence. Findings also support the importance of achievement goals collectively in linking motivational reasons to engagement outcomes. Among the three goals, self-based goals play the most salient role in effort and persistence strengthening the benefits of autonomous motivation for effort and persistence.

Students who engage in mathematics learning due to interest and personal meaning exert greater effort and persistence in the subject. The benefit of autonomous motivation for effort and persistence was strengthened by adopting self-based goals, suggesting that students who channel their interests in mathematics by striving to improve their past performance, or avoiding doing worse than before, are more likely to persevere when faced with challenges. Although controlled motivation did not directly predict effort and persistence, its indirect effect through self-based goals was positive, suggesting that when channelled through pursuing self-based goals, controlled motivation can be the reason facilitating effort and persistence. Thus, regardless of their motivational reasons, when students strive to pursue self-based goals, they would exert greater effort and persist at tasks because they know that surpassing their own previous performance or not performing worse than before is something achievable, and this gives them the confidence to persevere in being more proficient in mathematics.

The present findings hold practical implications for teaching, parenting, and school psychology practices seeking to foster student engagement in Singapore mathematics classrooms and those in East Asia, characterised by high competitiveness and school achievement orientation. Considering the benefits of autonomous motivation on effort and persistence, there is value in fostering autonomy-supportive practices among mathematics teachers. In this regard, Reeve (2016) asserts that teachers could provide rationale of a task in informational and non-controlling language to help students make motivational transition from perceiving that the task is not worth carrying out to something that is worth doing. Teachers could also provide information on how students they could progress in mastery of concepts and analysis of mathematics tasks. This would convince students that it is within their ability to take ownership of their progress in learning.

References

Cai, E. Y. L., & Liem, G. A. D. (2017). 'Why do I study and what do I want to achieve by studying?' Understanding the reasons and the aims of student engagement. *School Psychology International, 38*(2), 131–148. https://doi.org/10.1177/0143034316686399

Elliot, A. J. (1999). Approach and avoidance motivation and achievement goals. *Educational Psychologist*, *34*(3), 169–189. doi:10.1207/s15326985ep3403_3.

Elliot, A. J., Murayama, K., & Pekrun, R. (2011). A 3 x 2 achievement goal model. *Journal of Educational Psychology*, *103*(3), 632–648. doi:10.1037/a0023952.

Hogan, D., Chan, M., Rahim, R., Kwek, D., Maung Aye, K., Loo, S. C., Sheng, Y. Z., ... Luo, W. (2013). Assessment and the logic of instructional practice in secondary 3 English and mathematics classrooms in Singapore. *Review of Education*, *1*, 57–106. doi:10.1002/rev3.3002.

Johnson, M. L., & Kestler, J. L. (2013). Achievement goals of traditional and nontraditional aged college students: Using the 3× 2 achievement goal framework. *International Journal of Educational Research,* 61, 48-59.

Kaur, B., & Ghani, M. (2012). Learning experiences of low attainers. In B. Kaur, & M. Ghani (Eds.), *Low attainers in primary mathematics* (pp. 133–158). Singapore: World Scientific Press.

Lavigne, G. L., Vallerand, R. J., & Miquelon, P. (2007). A motivational model of persistence in science education: A self-determination theory approach. *European Journal of Psychology of Education*, *22*(3), 351–369. doi:10.1007/BF03173432.

Marsh, H. W., Hau, K. T., Artelt, C., Baumert, J., & Peschar, J. L. (2006). OECD's brief self-report measure of educational psychology's most useful affective constructs: Cross-cultural, psychometric comparisons across 25 countries. *International Journal of Testing*, *6*(4), 311–360. doi:10.1207/s15327574ijt0604_1.

Michou, A., Vansteenkiste, M., Mouratidis, A., & Lens, W. (2014). Enriching the hierarchical model of achievement motivation: Autonomous and controlling reasons underlying achievement goals. *British Journal of Educational Psychology*, *84*(4), 650–666. doi:10.1111/bjep.12055.

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 International Results in Mathematics and Science.* Retrieved from Boston College, TIMSS & PIRLS International Study Center website: https://timssandpirls.bc.edu/timss2019/international-results/

Preacher, K. J., & Hayes, A. F. (2008). Asymptotic and resampling strategies for assessing and comparing indirect effects in multiple mediator models. *Behavioral Research Methods*, *40*(3), 879–891. doi:10.3758/BRM.40.3.879.

Reeve, J. (2016). Autonomy-supportive teaching: What it is, how to do it. In W. C. Liu, J. C. K. Wang, & R. Ryan (Eds.), *Building autonomous learners* (pp. 129–152). Singapore: Springer.

Ryan, R. M., & Connell, J. P. (1989). Perceived locus of causality and internalization: Examining reasons for acting in two domains. *Journal of Personality and Social Psychology*, *57*(5), 749–761. doi:10.1037/0022-3514.57.5.749.

Ryan, R. M., & Deci, E. L. (2000). Intrinsic and extrinsic motivations: Classic definitions and new directions. *Contemporary Educational Psychology*, *25*(1), 54–67. doi:10.1006/ceps.1999.1020.

Vansteenkiste, M., Lens, W., Elliot, A. J., Soenens, B., & Mouratidis, A. (2014). Moving the achievement goal approach one step forward: Toward a systematic examination of the autonomous and controlled reasons underlying achievement goals. *Educational Psychologist*, *49*(3), 153–174. doi:10.1080/00461520.2014.928598.

Vansteenkiste, M., Smeets, S., Soenens, B., Lens, W., Matos, L., & Deci, E. L. (2010). Autonomous and controlled regulation of performance-approach goals: Their relations to perfectionism and educational outcomes. *Motivation and Emotion*, *34*, 333–353. doi:10.1007/s11031-010-9188-3.

STUDENTS' PERSPECTIVES ON TEACHING BEHAVIORS TO ENHANCE THEIR MOTIVATION FOR PARTICIPATION IN MATHEMATICS CLASS COMMUNICATION

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This study explores the factors contributing to the teaching behaviours that can enhance students' motivation to participate in classroom communication, from the perspective of students. A questionnaire, developed based on a year-long qualitative investigation, was administered to 542 junior high school students. Exploratory factor analysis was applied separately to three communication-related learning activities: listening to the lecture, asking questions, and discussing with peers. Eight factors relating to three facets — students' cognitive needs in mathematics, a safe environment and good atmosphere, and teachers' arrangements of modes and materials for activities — are identified. The study also revealed the inevitable intertwining between cognitive and affective facets from the factors identified.

INTRODUCTION

Taiwan is recognized for its high achievements in international comparison studies such as Programme for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS); however, it faces challenges with students' low affective performance (OECD, 2023; Mullis et al., 2020). For instance, only 23% of 8th graders in TIMSS 2015 reported high engagement in mathematics teaching, lower than the international average of 43% (Mullis et al, 2016). Additionally, as many as 56% of 8th graders in TIMSS 2019 reported a dislike for learning mathematics, higher than the international mean of 41% (Mullis et al, 2020). Addressing the issue of student motivation in mathematics learning is a significant challenge for Taiwan. Furthermore, these statistics indicate that this is a global issue requiring attention.

One aspect of students' engagement in mathematics class is their participation in communication within class. Enhancing students' communication in mathematics class not only assists them in engaging in mathematical thinking and understanding, but also helps them cultivate critical competence required for the 21st century (Ananiadou & Claro, 2009; Xu & Clarke, 2019). Niss (2003) indicated two critical facets of mathematics communication: (1) understanding others' written, visual or oral texts, and (2) expressing oneself in oral, visual or written form. In the present study, our focus is on enhancing students' oral communication in mathematics class, which includes listening to teachers' lecture, asking questions, and discussing with peers. However, we are aware that other forms of media are also involved in this communication.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 120–127). PME.

Adopting the aspect of the student-centred teaching approach, this study explored the following research questions related to students' oral communication in mathematics class by investigating students' perspectives,

1. What factors contribute to the teaching behaviors that promote students' listening to teachers' lecture in mathematics class?
2. What factors contribute to the teaching behaviors that promote students' asking questions in mathematics class?
3. What factors contribute to the teaching behaviors that promote students' discussing with peers in mathematics class?

CONCEPTUAL FRAMEWORK

As illustrated in Figure 1, students' oral communication in mathematics class involves two key participants — the teacher and their peers. Following Niss (2003), we focused on students' understanding of others and expressing themselves. Therefore, in terms of enhancing communication between students and the teacher, we examine which teaching behaviours can increase students' motivation to listen to the teacher's lecture and to ask questions. Listening to the lecture and asking questions are commonly observed interaction methods in traditional Chinese mathematics class (Shao et al., 2013). Regarding enhancing communication among students, we explore teaching behaviours that can encourage peer discussions. Student discussions have been emphasized and promoted in several recent curriculum reforms in Taiwan influenced by Western educational practice (Wang, & Hsieh, 2017).

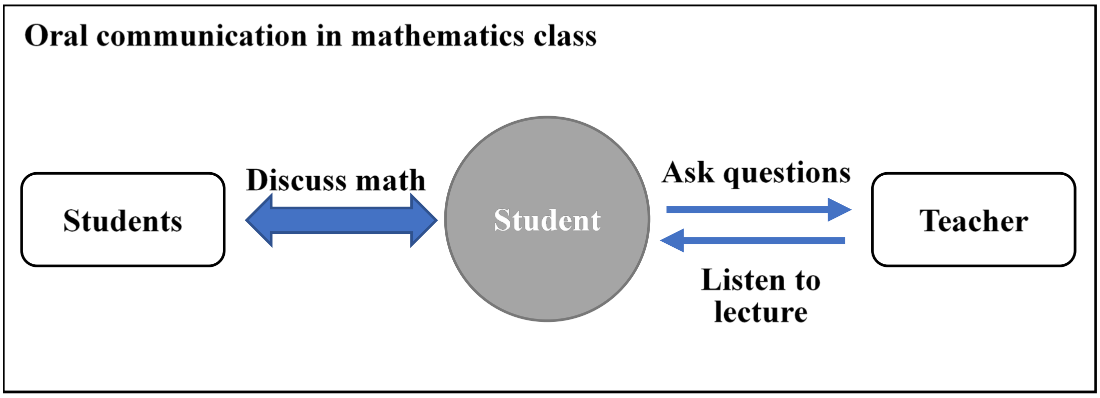

Figure 1: Conceptual framework of this study

RESEARCH METHOD

Design and Instrument

The study was conducted in two stages. In the first stage, a year-long investigation was conducted in the first author's junior high school class, consisting of 30 students, using question-oriented mathematical diary writing. During this stage, the students responded to 198 open-ended questions across 68 diary entries, aiming at gathering their opinions on the teaching behaviours that could enhance their mathematics learning motivation. Through content analysis of the students' responses and a literature review, 139 teaching behaviours in 11 different learning activities (e.g, asking questions, discussion, and doing math homework) were identified (Chiang et al., 2023). The teaching behaviours were then used to develop 6-point Likert scale (ranging from strongly disagree to strongly agree) items of 11 questions for the second stage.

The present paper used three questions relating to student oral communication in mathematics class: (1) Would the following situations make you more willing to listen to the lecture during math class? (2) Would the following situations make you more willing to ask questions during math class? (3) Would the following situations make you more willing to discuss mathematics with your classmates? The three questions comprised 16, 18, and 15 items, respectively.

Participants

The participants included 542 students from 6 junior high schools in Taiwan. In each school, one class was randomly selected from each of the 7th, 8th, and 9th grades. The sample consisted of 32.7% of 7th graders, 35.2% of 8th graders, and 32.1% of 9th graders, respectively.

Data Analysis

For each item, the 6-point Likert scale, which includes strongly disagree, disagree, a little disagree, a little agree, agree, and strongly agree, was converted to 1 to 6 points. Exploratory factor analysis (EFA) was performed on students' responses to determine the factors contributing to the teaching behaviours that can enhance students' motivation to participate in oral communications in class. EFA, using principal component extraction method with varimax rotation method was individually performed on each of the three learning activities: listening to the lecture, asking questions, and discussing mathematics with peers. Kaiser-Meyer-Olkin tests revealed that the sample was adequate for EFA (0.931~0.953). Eigenvalues were used to determine the number of factors (those with values exceeding 1). The items with factor loadings' absolute values less than 0.4 were deleted, and the remaining items were used to rerun EFA. The average scores of each item and each factor were then calculated. Subsequently, paired *t*-tests were performed to assess differences in students' endorsements across factors, while one-sample *t*-tests were utilized to compare these endorsements to the neutral midpoint of 3.5.

RESEARCH FINDINGS

Listening to the lecture

The final run of EFA was performed on 15 items regarding the teaching behaviours to enhance students' motivation to listen to the lecture, resulting in two factors that explains a total variance of 57.0% (see Table 1). The first factor, *mathematical understanding and equality*, comprised 8 items with factor loadings ranging from 0.422 to 0.839. This factor simultaneously involved the teaching behaviours that facilitate students' understanding, such as the providing clear explanation (M606), and informing students on how to apply what they have learnt to solve problems (M610), and that reflect the equal treatment of students, such as the teacher's attentive care for each student's learning condition (M614) and the assurance of fairness in the classroom (M615). The second factor, *teaching device and atmosphere*, comprised 5 items with factor loadings ranging from 0.572 to 0.764. This factor represented a group of teaching behaviours that use specific methods or create an enjoyable atmosphere to augment oral lectures, for instance, employing games or magic to illustrate mathematical concepts (M601), and fostering a relaxed classroom atmosphere (M607).

The average scores of the first and the second factors are 4.9 and 5.0 respectively, with item score ranges for the first and second factors being 4.0 to 5.2 and 4.7 to 5.3. The average scores for both factors are significantly higher than the neutral point 3.5 ($p < .01$), indicating students' strong endorsement of the teaching behaviours in both factors. This suggests that students would be more willing to listen to the teacher's lecture if their teachers implement the teaching behaviours that address both their mathematical understanding and equality, and when their teachers integrate the oral lecture with intriguing teaching activities and a positive atmosphere.

Listen to the teacher's lecture	Loading	Mean	SD
Mathematical understanding and equality		**4.8**	**0.8**
M610 When the teacher informs us about how the learned content can be applied in problem-solving.	0.839	5.0	1.1
M606 When the teacher's explanations are clear and easy to understand.	0.654	5.2	1.0
M615 If the teacher avoids favouritism towards male or female students during class and ensures fairness.	0.633	5.1	1.1
Teaching device and atmosphere		**5.0**	**0.8**
M601 When the teacher employs engaging activities to explain mathematical concepts, such as games, magic tricks, storytelling, etc.	0.764	5.3	1.0
M603 When the teacher uses computers, tablets, or similar devices for explanations.	0.721	4.7	1.3
M607 When the classroom atmosphere is relaxed and enjoyable.	0.577	5.3	0.9

Table 1: Statistics of factors contributing to teaching behaviours enhancing listening to lecture and exemplified items.

Asking questions

The final run of EFA was performed on 18 items regarding the teaching behaviours to enhance students' motivation to ask questions, resulting in four factors that explains a total variance of 63.4% (see Table2). The first factor, *intellectual needs*, includes five items with factor loadings ranging from 0.579 to 0.795. This factor represents teaching behaviours that necessitate the fulfillment of students' intellectual needs. This occurs in situations such as when tasks are challenging (M203) or the content is important (M204), and future learning will be affected (M204) or confusion will remain (M207) if questions are not raised. The second factor, *safe environment*, comprises five items with factor loadings ranging from 0.565 to 0.804. This factor signifies teaching behaviours that create a safe learning environment, including not being interrupted (M211) or laughed at (M210) even when asking a simple question, and teacher's caring for students' learning conditions (M216) in a manner that is like a friend (M217). The third factor, *classroom atmosphere*, includes five items with factor loadings ranging from 0.538 to 0.748. This factor represents teaching behaviours related to the creating joyful classroom atmosphere, including teacher's demonstration of a cheerful mood (M202), giving praise or gifts to students who ask questions to make the classroom atmosphere relaxed and enjoyable (M205, M208, M212). The fourth factor, *teacher's responding to questions*, consists of three items with factor loadings ranging from 0.515 to 0.815. This factor indicates how teachers respond to students' questions, allowing students to ask any questions at any time (M214), and ensuring that questions get answered (M215); however, the factor also involves teacher's displaying angry if students have questions but do not ask them (M218).

The average scores for factors one through four are 4.6, 4.9, 4.7, and 4.1, respectively. The item scores for factors one through four range from 4.4 to 4.8, 4.8 to 5.1, 4.5 to 4.9, and 3.6 to 4.8, respectively. The average scores for all four factors are significantly higher than the neutral point of 3.5 ($p < .01$), indicating students' endorsement of these teaching behaviours as encouraging them to ask questions. Paired t-tests revealed significant differences between each pair of these four scores ($p = .00$). The high score of the second factor suggests the importance of creating a *safe environment* for asking questions if a teacher expects this from their students. In addition, the results highlight the priority for a teacher to arrange their teaching behaviours if they want to create a classroom where students feel encouraged to ask questions.

Asking questions		Loading	Mean	SD
Intellectual needs			**4.6**	**0.9**
M203	When the questions are a bit challenging and provide a sense of difficulty for me.	0.795	4.4	1.3
M204	When I recognize that the content being learned is crucial, and not understanding it might affect my subsequent learning.	0.786	4.8	1.2
Safe environment			**4.9**	**0.9**
M211	If neither the teacher nor classmates mock anyone for asking questions, regardless of whether the question is simple or only the asker is uncertain about it.	0.804	4.9	1.2

M217	If the teacher interacts with us in a friendly manner, akin to a friend.	0.614	5.1	1.1
Classroom atmosphere			**4.7**	**0.9**
M208	If asking questions is rewarded with extra credit or small gifts.	0.748	4.5	1.4
M205	When the classroom atmosphere is relaxed and enjoyable.	0.609	4.8	1.1
Teacher's responding to questions			**4.1**	**1.0**
M214	If the teacher encourages us to ask questions at any time, even if it interrupts the teaching.	0.815	4.1	1.4
M215	If the teacher consistently answers any questions posed by students.	0.515	4.8	1.2

Table 2: Statistics of factors contributing to teaching behaviours enhancing asking questions and exemplified items.

Discussing with peers

The final run of EFA was performed on 15 items regarding the teaching behaviours to enhance students' motivation to discussing with peers, resulting in two factors that explains a total variance of 62.1% (see Table 3). The first factor, *curiosity*, comprises eight items with factor loadings ranging from 0.667 to 0.847. This factor involves teaching behaviours that arouse students' curiosity from both cognitive and affective aspects. For example, when questions for discussion are challenging (M401) or interesting (M402), the thinking process they arouse is intriguing (M415), and their solutions evoke curiosity to find out (M402). The second factor, *mode and material of activities*, includes seven items with factor loadings ranging from 0.463 to 0.764. This factor represents teaching behaviours that involve the use of specific modes or materials to enhance student discussions, such as arranging small group activities with friends, group competitions, and buzzing in (M403, M404, M411), as well as incorporating mathematical magic, puzzles, and real-life topics into teaching materials (M405, M410).

The average scores for the first and second factors are 5.0 and 4.8, respectively, with item scores ranging from 4.9 to 5.2 for the first factor and 3.8 to 5.2 for the second factor. Both the average scores significantly surpass the neutral point of 3.5 ($p < .01$), indicating students' endorsement of the teaching behaviours related to these two factors to encourage their discussion in class. The results suggest that teachers could address students' curiosity from both cognitive and affective aspects, and provide feasible arrangements of modes and materials to engage students in mathematical discussions.

Discussing with peers		Loading	Mean	SD
Curiosity			**5.0**	**0.9**
M412	If discussions can clarify my doubts and reveal the underlying principles.	0.847	5.1	1.0
M401	When the topic under discussion is somewhat challenging and difficult.	0.742	4.9	1.2
M402	When the scenarios for discussion are designed to be intriguing, and the answers arouse curiosity.	0.687	5.2	1.0
Mode and material of activities			**4.9**	**0.8**
M404	When we are seated in groups.	0.764	5.0	1.2
M403	When there are competitive activities such as quizzes or group competitions during class.	0.753	5.0	1.2
M405	When class incorporates intellectually stimulating activities like puzzle games or mathematical magic.	0.687	5.2	1.1

Table 3: Statistics of factors contributing to teaching behaviours enhancing discussing with peers and exemplified items.

CONCLUSION

The factors contributing to the three learning activities (listening to the lecture, asking questions, discussing with peers) can be categorized into three groups. The first group involves students' pursuit of mathematical understanding; they are willing to engage in class communication when they have cognitive needs in mathematics. The second group is related to safe environments and a positive atmosphere, which pertain to students' sense of safety in class. In East Asian countries, where collectivist culture prevails, especially in Taiwan (Hofstede, 1986), it has been reported that students are particularly afraid of failure (OCED, 2018). Therefore, providing students with a sense of safety in mathematics classes is especially important if oral communication is expected. The third group involves teachers' arrangement of modes and materials for activities. Our findings propose several feasible approaches that are endorsed by students. Furthermore, the study also identified the intertwining of cognitive and affective facets in several factors, echoing the long-standing importance of integrating these two aspects in mathematics instruction (McLeod, 1992). However, this integration may not have been well implemented in mathematics classes (Wang et al., 2023).

References

Ananiadou, K., & Claro, M. (2009). *21St century skills and competences for new millennium learners in OECD countries*. OECD education working papers, no. 41. OECD.

Chiang, C.-M,. Wang, T.-Y, & Hsieh, F.-J .(2023). A long-term investigation on junior high school students' prspectives on what teachers should do to enhance their mathematic learning motivation. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, M. Tabach(Eds*.), Proc.*

46th Conf. of the Int. Group for the Psychology of Mathematics Education (Vol. 1, pp. 368), PME.

Shao, G., Fan, Y., Huang, R., Ding, E., & Li, Y. (2013). Mathematics classroom instruction in China viewed from a historical perspective. In Li, Y.-P, & Huang, R.-J.(Eds*.), How Chinese teach mathematics and improve teaching* (pp. 11-28). Routledge.

Hofstede, G. (1986). Cultural differences in teaching and learning. *International Journal of intercultural relations*, *10*(3), 301-320.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, (pp.575-596). Macmillan.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International Results in Mathematics*. Retrieved from Boston College. TIMSS & PIRLS International Study Center.

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 International Results in Mathematics and Science*. Retrieved from Boston College. TIMSS & PIRLS International Study Center.

Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis & S. Papastravidis (Eds.), *3rd Mediterranean conference on mathematics education* (pp. 115–124). Hellenic Mathematical Society.

OECD (2019). *PISA 2018 results (volume III): What school life means for students' lives*. OECD.

OECD (2023), *PISA 2022 Results (Volume I): The State of Learning and Equity in Education*, OECD.

Wang, T.-Y., & Hsieh, F.-J. (2017). Taiwanese high school students' perspectives on effective mathematics teaching behaviors. *Studies in Educational Evaluation*, *55*, 35-45.

Wang, T.-Y., Yang K.-L., Lin F.-L. (2023). Exploring the Feasibility of Co-construction Among Mathematics Teachers and Teacher Educators: Analysis of Discourse in a Product-based Teacher Professional Development Program. *Mathematics Teacher Education and Development*, Advance online publication. https://mted.merga.net.au/index.php/mted/issue/view/53

Xu, L., & Clarke, D. (2019). Speaking or not speaking as a cultural practice: Analysis of mathematics classroom discourse in Shanghai, Seoul, and Melbourne. *Educational Studies in Mathematics, 102*, 127-146.

MENSURATING THE AREA OF A STOLEN LAKE: MATHEMATISING A HISTORICAL EVENT

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In this research report, students mathematise the image of a drained lake and reflect upon the repercussions of the draining, specifically in terms of an Indigenous nation which had lived beside the lake from time immemorial. Using Google Earth, students explore the notion of area by comparing the lake with a personal landmark in their locality. The study explores how students describe area in terms of new mathematical relations. From their descriptions, six themes emerged that enrich the conceptions of area.

INTRODUCTION

In this research report, I explore grade eight (13 year-olds) students' contemplation and consideration of the draining of a lake using the mathematical concept of area. Area continues to be a challenging topic for students even as late as grade 9 (Lehmann, 2023; van de Walle et al., 2019). It is usually conceived of as a formula rather than a comparison to a standard. In this study, students engage in a socio-political issue, namely the draining of a lake in 1924 by the local government that displaced Indigenous people who had lived beside the lake from time immemorial. I approach the mathematical notion of area as one way to understand the significance of the draining. Rather than thinking of area as a bite-sized concept to be applied to a socio-political topic, this study presents an approach that explores the draining while thinking mathematically. Area is approached as a form of mathematising, and it is seen as a positioning of worldly phenomena into a commensurable perspective.

This study draws on the theoretical work of Reyes (2022), who describes mathematical concepts as developing through translating worldly relations into commensurable symbolic forms and reconfiguring those forms into new relations. In this study, I take the position that this is how students learn mathematics. I take a critical approach to teaching mathematics (Skovsmose, 2011) and align with Dominguez et al. (2023), who suggest that learning is not about student changing to understand the concept. Rather, they explore how mathematics can be reconceived so that it is accessible to students.

In this study, students engage with thinking about a socio-political issue using an online digital mathematical tool, *Google Earth*, and are prompted to articulate ways of relating to the historical event. One overarching question the study asks is how to make the mathematical concept of area more meaningful for students so that they actually see why they might need to know it. Also: how it can be presented and explained in a way that the focus is not on calculation but on relations. The study seeks to understand

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 128–135). PME.

student expression of meaning about a socio-political issue by exploring the notion of area.

THEORETICAL FRAMEWORK

The framework upon which I ground this study is the notion of mathematising, which originally emerged from the work of Freudenthal (1973). He established Realistic Mathematics Education (RME) in the early 1980s and which has since become a Dutch mathematics education tradition. RME was founded on mathematising, whereby the goal was to connect mathematics to reality as closely as possible and to draw on common sense and informal mathematics as a starting point for students and in which mathematics can be elicited from everyday life situations (Heuvel-Panhuizen, 2000). Wheeler (2001) similarly suggested that mathematising is the act of producing "mathematics…in situations where something not obviously mathematical is being converted to something that is" (p. 51). Both Freudenthal and Wheeler describe mathematising as engaging with the world in non-formulaic, creative, and innovative ways that draw on mathematical relations. For them, mathematics emerges from mathematisations which engage with the world.

Reyes (2022) furthers the practice of mathematising within his theoretical framing of how the discipline of mathematics develops. For Reyes, mathematics does not emerge from the blinding light of timeless truth, but from the translations of worldly relations into commensurable symbolic forms, which can then be reconfigured via inventive practices, leading to the articulation of new relations. Mathematising for Reyes is the act of taking world relations, such as the size of a lake, converting them into symbolic forms, and reconfiguring those symbols into new relations. When more relations are forged, the mathematics inherent within that relation becomes "more semiotically dense, more resistant to challenge, and more real" (p. 56). For Reyes, manipulating symbols is meaningless without strong relations. However, when those symbols represent something local, something that touches one's own personal experience, new meanings and relations develop. These new meanings are not only about taking in facts and piecing them together cognitively, but also are about linking to one's sense of such things as personal commitments, historical awareness, aesthetics, and ethics.

Barsalou (2020), a psychologist and cognitive scientist, aligns with Reyes' way of thinking by positing a grounded cognitive theory whereby understanding a concept is not learning an abstraction but developing a "competence or disposition for generating infinite conceptions of a category" (p. 9). An example of this, cited by Reyes, is Thurston's seven meanings of a derivative. "The meaningfulness of 'derivative', according to Thurston, comes not from each particular definition but from its unlimited polysemy" (in Reyes, p. 6). In the same way, when students create new ways of expressing area by contemplating Lake Sumas as stolen from the Sumas Indigenous people, area becomes meaningful and significant through its polysemy.

My research explores what new relations emerge from students' contemplation of this event as they use area to mathematise the size of the lake. Specifically, my research question is what new relations of area do students describe when contemplating the draining of Lake Sumas?

METHODS

The data in this study was collected in December 2023. The researcher who developed the activity taught two grade eight classes (13 year-olds) in a school in Western Canada. One class had 19 students, the other 22; in both classes, the students sat in pairs. The grade 8s were informed of the study by the classroom teacher before the researcher taught the class; all students agreed to participate in the study. There was no video or audio recording; however, there was a research assistant who took notes. All of the data was taken from a worksheet that students were given at the beginning of class and which was referred to as the lesson progressed. Students had access to *Google Earth* on a laptop and were given time during the class to complete portions of the worksheet. The curriculum in the province from which the researcher taught focuses on competencies as central to mathematics learning rather than content. That is, as examples, there is a focus on graphing and/or reasoning as opposed to learning how to complete the square or how to calculate an area. One competency relevant to this study is the ability to “engage in problem-solving experiences that are connected to place, story, cultural practices, and perspectives relevant to local First Peoples’ communities, the local community, and other cultures” (BCMoE, 2015). This competency is the result of the Truth and Reconciliation Commission of Canada (TRC, 2015) based on Canada’s mistreatment of Indigenous people. In the curriculum, area is introduced in grades 5 and 6 in a way that focuses on using grid paper to calculate area and determining the area of complex shapes. Surface area, as a topic, is introduced in grade 8. The activity in this study is an introduction to surface area. While the activity focuses more on the area on a 2D map than on the surface area on a 3D object, the activity is a move from area as calculation to area as proportion and toward thinking about surface area. Some of the worksheet questions are: How large of an area is 36 km^2? How could you put 36 km^2 into perspective for someone who doesn't know what km^2 means? Looking at Lake Sumas, what do you notice? (An image of the lake before 1924 is projected on a screen for the students to see), What do you wonder? Is Lake Sumas a big lake? What would you like to know to help you determine whether it's a big lake or not? Drag the lake and position it over top of a landmark you are familiar with, such as your home, school, or nearby park. Please sketch the image of this overlay below. Why did you choose that landmark?

All responses were typed into a document and organized by question. These responses were then examined, and themes were created. The themes were developed by specifically focusing on expressions of area that were grounded in students’ own personal meaning making but yet also connected to the story of Lake Sumas.

Lake Sumas story

I base the lesson on a story of the Sumas region in British Columbia. Sumas Prairie lies approximately 100 kilometres east of Vancouver and is a large fertile area with many homes and farms. Much of the dairy and poultry in grocery stores in Vancouver come from the area. In November 2021, an atmospheric river dropped a significant amount of rain within a short period of time leading to floods that damaged houses, farms, and livestock. One of British Columbia's main highways, which passes through the area, was blocked. Usually, when I ask students what they remember from this event, it's the empty shelves in the local supermarkets.

What student typically do not know is that the area had previously been a lake lived on by the Sumas Indigenous nation. For the Sumas people, 85% of their diet came from the lake and there is evidence that they used the lake as far back as 400 BCE. Many of the stories they shared throughout generations were based on the lake. But in 1924 the lake was drained. The rivers feeding into the area were redirected, and a pump station was built so the land could be used for farming. The Sumas people were not consulted about the project, even though they had land claims to regions of the Sumas valley. One Sumas elder said of the 1924 drainage, "They choked the lake".

My interest was to mathematise the lake with respect to its size so that students get a sense of how significant its draining was on both the environment and the Sumas people. The students use *Google Earth* to draw a boundary around the lake as it was pre-1924 (I have images I show the students). Once the boundary is drawn, *Google Earth* calculates the area and the perimeter; students do not calculate these values. Lake Sumas was approximately 36 square kilometres. It is challenging to notice that the lake is very large as the area and perimeter, provided by *Google Earth* are numbers without a story, and also because Lake Sumas is situated between two mountains which makes the lake appear small. However, I ask students to drag the bounded area to another part of the map. I want them to compare the size of Lake Sumas with a landmark they are more familiar with to link the significance of the draining with a landmark they find meaningful.

DATA AND ANALYSIS

The research question asks how students describe area. In this section I share students' responses from the worksheet. In particular, I am paying attention to how students are describing area using new relations based on the draining of Lake Sumas. [Note that I have not corrected the grammar or spelling of the student responses; this was a choice to minimize the distractions of many needed corrections.]

Theme 1: Personal activities in relation to Lake Sumas

There were some instances of algorithmic or rule-oriented responses. There were seven out of 41 of these responses, of which I share two below, simply as contrast to what follows.

> Student_1: To get something that is squared, you have to multiply it by itself, meaning multiplying 36 by 36, which is 1296 km, and that sounds pretty large.
>
> Student_2: I think 36 km^2 is 1,296 kilometers is a form of measurement. Two is the number of times itself.

Both of the responses above are mathematically incorrect, but more importantly they are examples of non-personal algorithmic approaches to area. That is, there seems to be no personal connection with what is written.

I contrast the above with the following in which 15 of the 41 students described area in terms of a personal activity. I share two here:

> Student_3: Well lake Sumas is big because if you were going to kayake across the lake it would take a long time to cross.

> Student_4: For me, 36 kilometers squared is an eight hour walk.

These two examples are not precise, but they do represent a generative conception of area in that area is related to an activity the student can do. The personal makes it meaningful but also size is being thought of temporally and through effort and energy.

Theme 2: Expressing agency in terms of Lake Sumas

In the following two statements, it is worthwhile to know that Abbotsford is a small city close to the Sumas region:

> Student_5: It doesn't look big on the map but if you think about it, it does actually look bigger than Abbotsford which is quite big.

> Student_6: If you ever drive to Abbotsford, you can see how much land it is.

These responses are less about doing an activity, as in theme 1, and more about the willingness to take the time to consider and reflect upon the size of the lake. "If you think about it" or "if you ever drive" position the student as having the agency to choose to act in a reflective way. In these cases, mathematics is about action one can take if one wants to get a better sense of what is going on.

Theme 3: Comparing other landmarks with Lake Sumas

Before prompting the students to drag Lake Sumas and compare it with a landmark of their choosing, many students had described the size of Lake Sumas by comparing it to Cultus Lake. Cultus Lake is very close to the Sumas region and is usually visible when looking at Lake Sumas on *Google Earth*. Cultus Lake is a popular summer spot some students have visited. It is considered a large lake and yet is approximately five times smaller than Lake Sumas. 14 out of 41 related the size of Lake Sumas to Cultus Lake.

> Student_7: Yes, it's a big lake. You can tell because in comparison, it's much bigger than Cultus lake.

When I project the image of Lake Sumas before draining in 1924 on a projector screen, the lake is transparent and the farm lands that exist now can be seen.

> Student_8: There is farm lands in the lake, and the farmland is big. There were maybe 100-200 for a lake it's big.

In both of the above cases, it is noteworthy that students are comparing Lake Sumas with another geographical landmark that can be seen when looking at *Google Earth*. They are choosing something close by and accessible and seeing area as comparison rather than as number.

Theme 4: Relating the size of Lake Sumas to the mathematical notion of average

Student_9: I would like to know the sizes of some big lakes and small lakes, to see where lake sumas would fall.

Student_10: Lake Sumas is a pretty big lake in my opinion but there are smaller and bigger lakes in the world so I wouldn't say it's one of the biggest.

In this theme, students appeal to another mathematical topic: average. They position Lake Sumas, not in comparison with another single landmark, but rather refer to the sizes of other lakes to get a better perspective of whether Lake Sumas is large.

Theme 5: Relating personal landmarks with Lake Sumas

In one question on the worksheet, I asked students to drag the outline of Lake Sumas and overlay it with a landmark of their choosing. I ask them to draw an image of this overlay. I present two examples in Figure 1 that were indicative of what most students drew. In general, in all these cases, there is a very personal appeal to a landmark that students consider important to them.

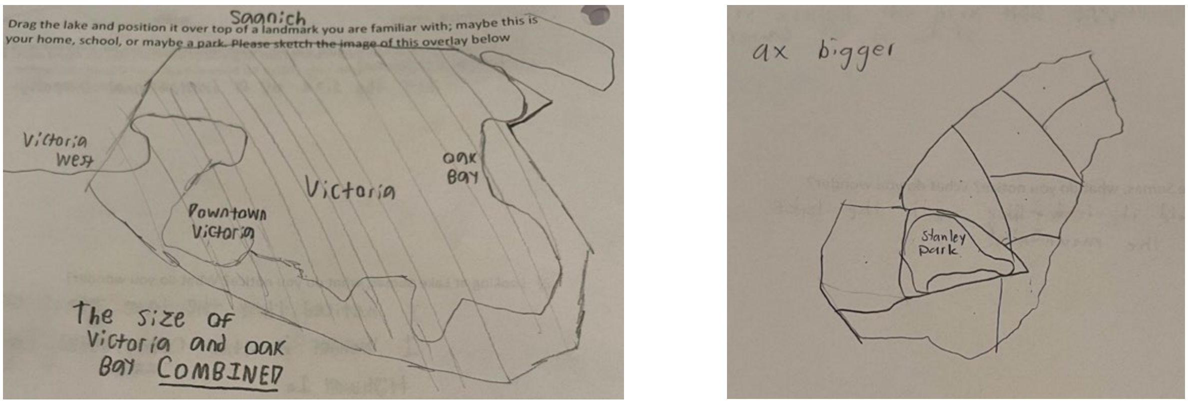

Fig. 1: a) Victoria and Oak Bay COMBINED; b) Nine time larger than Stanley Park

Figure 1a shows Lake Sumas covering the city of Victoria and Oak Bay, British Columbia. The capitalized word “COMBINED” emphasizes the student’s surprise of how much Lake Sumas covers. Figure 1b shows a dissection of Lake Sumas over Stanley Park, with Lake Sumas sketched and labelled to be “9X bigger” (9 times larger). The student who presented this particular sketch, overlaying Stanley Park, wrote that they chose this as their landmark because they had done “marine biology research” there with their father. 23 out of the 41 students gave reasons of why their landmark was meaningful (the remaining students left this question blank). Here are some examples of other students’ statements:

Student_11: Because I remember always going there with my family as a kid. It's nostalgic for me.

Student_12: Because my grandma is there.

Student_13: Because it's the building I go to for school everday, it's quite large and I'm pretty familiar with it.

The comparison of areas is made commensurable by overlaying the outline of Lake Sumas with one that has personal meaning. Students chose landmarks they were familiar with. There was a relevance to the landmark and with the overlay, that relevance was associated with Lake Sumas. When Lake Sumas is positioned in that frame, students can reflect on loss; that is, they might wonder what it would be like to lose their own landmark.

Theme 6: The size of Lake Sumas in terms of impact

Student_14: It's big because it impacted so many people.

Student_15: I think that it greatly impacted the coast-salish people that lived there, and it was basically their main source of food that was being removed, just so that the British Europeans could farm better.

Student_16: Well taking away the lake also took away the resources it gave to the Indigenous peoples (food, water, etc.).

In these responses, area no longer is perceived as a number or a comparison with another landmark, but instead is related to an impact on people's livelihoods. The size of the lake is a symbol of how people were treated when the lake was drained. In general, student recognized the travesty of the draining on the Indigenous people. In these responses, it is evident that the area of Lake Sumas is connected with the significance the draining had on the Sumas people.

CONCLUSION

In this research report, a historical event was shared with students that connected the draining of a lake with a devastating flood that occurred in 2021. Students use *Google Earth* to mathematise an image of the lake before it was drained by outlining the lake and dragging it to a landmark of their choice. This study focuses on how students describe the size of the lake in their various ways. The pedagogy outlined in the study aimed to present students with the view that mathematics is not always about algorithms but can also be conceived through a reconfiguration of worldly relations. Focusing on the area of the lake and trying to make sense of that size is not only a meaning making activity but also introduces new ways of relating to how the draining influenced the environment and the Sumas people. The research of this study aimed to identify some of the novel relations students used to relate to the size of the lake. In answering the research question, I identified six themes related to how area was conceived. These themes focused on area as related to: students' personal interests, such as kayaking; impact, whereby some students noted draining a large lake affects a lot of people; appealing to other mathematical topics, such as average; enacting agency, what students can do to improve their understanding of a particular situation; and personal landmarks that students can remember, visualize and visit. These themes are meaningful to the students since the students are making the connections themselves; but also, the themes contribute to the mathematical notion of area. The themes are not exhaustive; more studies will have to be undertaken to develop a more cohesive and robust sense of how students link mathematical topics with stories that involve history and cultural perspectives, specifically in regions where the curriculum calls for such links.

Area in this study is a way of seeing. It is a perspective rather than a formula or an exercise. This research contributes to the learning of mathematics when addressing socio-political issues in the classroom. As we think about the nature of mathematics, we might rethink what kinds of relations we want to develop when teaching mathematics.

References

Barsalou, L. W. (2020). Challenges and opportunities for grounding cognition. *Journal of Cognition*, *3*(1): 31, pp. 1–24. DOI: https://doi.org/10.5334/joc.116

BCMoE. (2015). *Math K–9: Curricular Competencies*. British Columbia Ministry of Education. (https://curriculum.gov.bc.ca/sites/curriculum.gov.bc.ca/files/curriculum/continuous-views/en_math_k- 9_curricular_competencies.pdf)

Dominguez, H., Abreu, S. & Peralta, M. (2023). Young philosophers: fifth-grade students animating the concept of space. *ZDM Mathematics Education*, 55, 1151–1171. https://doi.org/10.1007/s11858-023-01483-6

Freudenthal, H. (1973). *Mathematics as an educational task*. Reidel.

Lehmann, T. (2023). Learning to calculate surface area: a focus on strategy choice. *Research in Mathematics Education*, *25*:3, 301-322.

Reyes, G. M. (2022). *The Evolution of Mathematics: A Rhetorical Approach*. Pennsylvania State University Press, 2022.

Skovsmose, O. (2011). *An invitation to critical mathematics education*. Sense Publishers.

TRC (2015). *Honouring the truth, reconciling for the future: Summary of the final report of the Truth and Reconciliation Commission of Canada*. Truth and Reconciliation Commission of Canada.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2019). *Elementary and middle school mathematics: Teaching developmentally (10th ed.)*. Pearson.

Van den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour.* (pp. 1–32). Freudenthal Institute CD-rom for ICME 9.

Wheeler, D. (2001). Mathematisation as a pedagogical tool. *For the Learning of Mathematics*, *21*(2), 50–53.

SNAPSHOTS OF A TEACHER'S PRODUCTIVE TALK MOVES WHEN ORCHESTRATING A WHOLE-CLASS DISCUSSION

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Orchestrating productive mathematics discussions by building on students' ideas is challenging. Although certain talk moves involving eliciting student responses are associated with this high-leverage practice, they may not be sufficient for enhancing student reasoning. Telling, on the other hand, may play an important role despite the perception they are contradictory to a more interactive stance in teaching. In this paper, we examined how an elementary school teacher orchestrated a productive whole-class discussion through the skilful interweaving of talk moves and telling.

INTRODUCTION

Orchestrating productive mathematics discussions by building on students' ideas has been seen as a high-leverage teaching practice by many mathematics educators (e.g., Smith & Stein, 2011). This requires teachers to design tasks that reveal students' thinking, listen to, and build on students' ideas by noticing the mathematics in students' ideas (Choy, 2016). Consequently, teachers have adopted talk moves, such as eliciting ideas from students (Lobato et al., 2005), pressing students to clarify their thinking (Brodie, 2010), and revoicing ideas from other students (O'Connor & Michaels, 1993) to meaningfully incorporate students' ideas as part of their instructional practices. However, deploying these high-leverage moves in the classrooms may not lead to a productive engagement of students' thinking (Van Zoest et al., 2023). On the other hand, telling, which is often seen as contradictory to a more interactive stance in teaching, may have an important role to play when orchestrating discussions (Lobato et al., 2005). What is unclear is how the different talk moves can be used together with telling to engender a more productive mathematics whole-class discussion that enhance student reasoning to bring about conceptual understanding. In this paper, we aim to add to this conversation about the complexities of orchestrating discussions by providing snapshots of an elementary school teacher, Ms. Hannah, as she orchestrated a mathematically productive classroom discussion using a challenging problem on elapsed time across time zones. We frame this paper around the key research question: What are the different talk moves used by Ms. Hannah and how did she use them together with telling to bring about a productive mathematics whole-class discussion?

THEORETICAL CONSIDERATIONS

We begin by unpacking the idea of a productive mathematics discussion in terms of its patterns of discourse and examine how the different talk moves can support or hinder opportunities for students to reason mathematically. There are at least two main patterns of discourse that feature frequently in mathematics classrooms. First, as

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 136–143). PME.

described by Mehan (1979), the initiate-response-evaluate, or IRE, pattern of discourse is characterised by the teacher initiating talk by posing a question (I), which is followed by a response given by a student to the question posed (R), before teacher evaluates or gives a feedback to the response (E). In an IRE pattern of discourse, the questions asked are usually funnelling in nature, in which the teacher engages in most of the thinking by asking a series of questions that guide students through a procedure to a desired end (Herbal-Eisenmann & Breyfogle, 2005). In contrast, the second pattern of discourse—revoicing—as described by O'Connor and Michaels (1993), is usually initiated by a student's question or statement, which is rephrased by the teacher before inviting the student or others to explain or justify the initial question or statement. Such patterns of discourse are usually characterised by the use of focusing questions, which require teachers to listen to students' ideas (Rinaldi, 2001) and guide them based on what students are thinking (Herbal-Eisenmann & Breyfogle, 2005). With the aim of implementing a high cognitive-demand task to realise its design potential, it is crucial for teachers to adopt patterns of discourse that are more likely to engage students in thinking. For this study, we see revoicing as the pattern of discourse that is more aligned to the notion of productive mathematics discussions (Smith & Stein, 2011).

Orchestrating a productive mathematics discussion is challenging and so, it is useful to support teachers to do this ambitious work by providing a structure or routines of practice to frame their discussion. One such structure is the five practices—anticipating, monitoring, selecting, sequencing, and connecting—as proposed by Smith and Stein (2011). Here, we presuppose orchestrating such discussions as a deliberate practice and can be planned. The centrepiece of a productive discussion is a mathematics task that is designed to reveal students' thinking, as *anticipated* by the teacher. During the implementation of the task, the teacher will *monitor* the students' responses to the task, purposefully *select* and *sequence* students to present their work before the teacher guides students to *connect* the different responses to form a mathematical conclusion. Doing this involves talk moves such as eliciting ideas from students, asking for clarification of ideas, and teacher's revoicing of ideas to facilitate interaction (Van Zoest et al., 2023). Hence, these five practices place demands on a teacher's attention to listen with students and notice the mathematical aspects of students' ideas (Choy, 2016).

However, using some of these talk moves may sometimes be counterproductive. As highlighted by Van Zoest et al. (2023), eliciting ideas, asking to clarify ideas, and even revoicing, may "diminish opportunities to engage the whole class with a high-leverage contribution that is already available for discussion" (p. 251). For example, eliciting ideas from students may be counterproductive when students lack prior knowledge to construct new meanings among themselves. On the other hand, using a low-leverage practice such as telling may be productive when teachers are "describing a new concept or "summarising students work" that "inserts new information into the conversation" (Lobato et al., 2005, p. 110) in ways that promote conceptual growth. This brief discussion suggests that it not solely the types of talk moves teachers use, or even if teachers should tell students during classroom discussions; but more importantly, we need to acknowledge that telling together with other talk moves may form part of a “more sophisticated range of pedagogical actions” (Lobato et al., 2005, p. 131).

METHODS

The vignettes described in this paper were developed from data collected as part of a study that focused on developing a listening pedagogy (Rinaldi, 2001). The vignettes centred around Ms. Hannah (pseudonym), a mathematics teacher with more than 18 years of teaching experience at Quayside Primary School (pseudonym) in Singapore. As Ms. Hannah was familiar with the practices of orchestrating mathematics discussions (Smith & Stein, 2011), she was one of the teachers we followed closely in our study. In this paper, we will present snippets of how Ms. Hannah orchestrated mathematics discussions with a class of Grade 4 students around a challenging problem on elapsed time (Constance & Kelly, 2012) in the context of time zones. Grade 4 students in Singapore are expected to solve “problems involving time in 24-hour clock” (Ministry of Education-Singapore, 2012, p. 45) and they have encountered concepts related to the “awareness of time, succession, duration, and measurement of time” (Thomas et al., 2023, p. 55).

Data were generated from video recordings of the lesson, voice recordings of an interview with Ms. Hannah, photographs of students’ work during the lesson, and teaching artifacts such as the slides used for the lesson and other instructional materials developed by Ms. Hannah for this lesson. We parsed the video recording of the lesson and segmented the lesson into the different phases of the lesson. We then identified pedagogically significant moments related to Ms. Hannah’s use of the mathematics task and examined how she orchestrated the discussion through telling, eliciting information from the class, asking students to clarify their contributions, and inviting students to revoice their peers’ contributions (Van Zoest et al., 2023).

FINDINGS

Overview of the lesson

The centrepiece of the lesson is a problem in the context of time zones:

> At 11 30 in Singapore, a plane leaves for London. At what time in London will it be when the plane touches down if the duration of the flight is 13 hours? Explain your answer.

This is a challenging problem for Grade 4 students because it involves the concept of elapsed time and time comparison, which is difficult for many students (Constance & Kelly, 2012). In Singapore, students usually solve problems involving elapsed time in the contexts of tasks and journeys in the same time zone. Problems involving different time zones are considered challenging and are often given as problem solving tasks (e.g., see Chan, 2016, p. 184). As highlighted by Thomas et al. (2023), even students in the first year of secondary school (Grade 7) had difficulties understanding concepts of time comparison, such as durations and time zones, in a national assessment held in the state of Victoria, Australia.

Ms. Hannah started by recapping the ideas covered in the previous lesson on the 24-hour clock, time in seconds, and the use of timeline as a problem-solving tool (2 minutes). She then initiated a discussion around the notion of time comparison across different time zones (9 minutes) by relating to students' experiences of international travel and communication with overseas friends. The problem was introduced to the students, and they had some time to read and understand the question individually (2 minutes) before Ms. Hannah launched a short discussion to clarify any questions regarding the problem (4 minutes). Students were given 2 minutes to work on the problem on their own before they worked collaboratively to solve the problem on an A3-size working sheet (10 minutes). They were also prompted to provide alternative solutions if they could. As the students worked on the problem, Ms. Hannah moved around the class to monitor students' thinking and solutions before she launched a whole-class discussion by inviting selected groups of students to share their solutions and built on students' thinking to highlight key learning points (10 minutes). She then assigned questions from the workbook for students to work on while circulating the class to support students who had questions (15 minutes).

Vignette 1: Time Difference between Singapore and London

Ms. Hannah provided some time for her students to clarify any doubts about the question after the problem was introduced. Several students raised their hands. One of the students, S1, wanted to know if the time given was 11.30 a.m. or 11.30 p.m. Instead of answering the question, Ms. Hannah redirected the question to her students:

Ms. Hannah: So, he asked whether it is 11.30 a.m. or p.m. Anybody can answer [Student S1]? (Many students volunteered by raising their hands. Ms. Hannah picked Student S2 to answer)

Student S2: Morning.

Ms. Hannah: Morning. Why? Must explain [to your friend, S1].

Student S2: Because 11 30 is [written] in the 24-hour clock.

Ms. Hannah: (repeating what Student S2 said). It's in the 24-hour clock, right? Is it stated in the 24-hour clock? Am I right? (Many students nodded.). Okay. Any other questions?

One of the students, S3, noticed that the time difference between Singapore and London was missing and asked about it:

Student S3: What is the time difference between Singapore and London?

Ms. Hannah: Very good! This is a piece of critical information that is not there [referring to task], right? I am very happy that you saw that. That's one missing information in the task. Without that information, can you solve?

Whole class: No.

Ms. Hannah: Right. Without the piece of information, you can only solve part of it and not the whole thing, right? Anybody knows [what the time difference is]?

A student, S4, volunteered and Ms. Hannah signalled for S4 to answer. S4 gave the answer as 7 hours and Ms. Hannah confirmed the answer. She then asked S4 whether to "add or subtract 7 hours". To facilitate the discussion, Ms. Hannah wrote "Singapore: 08 00" on the board for S4 to consider.

Student S4: Minus seven hours?

Ms. Hannah: When [S4] said "minus seven hours", what would that be? Anybody? What would that be? (A few students raised their hands and S5 was picked to answer).

Student S5: Zero-one-hundred hours.

Ms. Hannah: (Repeating what S5 said) Zero-one-hundred hours (Wrote "London: 01 00" on the board). Very good. Can you see? What you say 7 hours difference, you also need to know whether it is earlier or later.

In these exchanges, we see how Ms. Hannah created opportunities to clarify S2's and S4's contributions. By doing so, she not only ensured that her students could see the ideas she wanted to highlight but also the reasoning behind the answers, which can be seen as case of productive clarifying (Van Zoest et al., 2023). There were also occasions where Ms. Hannah elicited key ideas from her students and then reaffirmed what the students said. On the surface, it may look like a standard Initiate-Response-Evaluate, or IRE pattern of discourse (Mehan, 1979), but it is really a case of productive telling (Lobato et al., 2005)—where Ms. Hannah explicitly directed the students to the essential ideas needed to solve the question. In this episode, S4, who knew the time difference between Singapore and London was able to provide that information without Ms Hannah doing so. A key distinction in Ms. Hannah's practice is that her telling was interweaved with the other talk moves to maintain student engagement in thinking about the problem.

Vignette 2: "Time plus duration?"

After the initial clarification, students actively worked on the problem individually before they worked collaboratively to solve the problem. Ms. Hannah monitored the different solutions worked out by the students and asked three groups of students to present their solutions. The first group worked out the solution in Singapore time before converting it to London time (see Method 1 in Figure 1):

Student S5: So, basically this is [S6]'s idea and so because Singapore time is 11 30, so she added 10 hours to twenty-one-thirty hours [sic]. And another three hours for the total duration of the flight to zero-zero-thirty hours in the morning. So, because London is seven hours behind Singapore, so zero-zero-thirty hours in the morning of Singapore, minus seven hours to seventeen-thirty in the evening at London.

Ms. Hannah: Okay. Can you just leave it there, please? (asked S5 to leave her answer there at the visualiser). Given them a round of applause. (Students clapped).

What did they do first? What was their first step? (A few students raised their hands, and she picked S7 to answer.)

Student S7: To add 13 hours.

Ms. Hannah: Add the duration of the flight. Agree? (Wrote on the whiteboard). Add the duration of the flight. Agree? (A few students said “yes”). So, the first thing they did: add the duration of fight. I’m just going to put “add”. Then what did they do next? (Picked S8)

Student S8: Minus seven hours.

Ms. Hannah: Why did they minus seven hours? What was that for?

Student S8: Because in London it’s seven hours… before Singapore time, so they have to minus seven hours to find the time in London.

Ms. Hannah: Okay. Convert to London time (wrote on the whiteboard). Yes? Agree? (Some “yes” from students). Anything you want to comment? Look at the working. It has to be clear. (S9 raised his hand and Ms. Hannah motioned for him to proceed).

Student S9: The eleven hundred hours plus 30 hours (sic) plus 13 hours… because both things are different, so I think you cannot just add them together (see highlighted text in Figure 1).

Ms. Hannah: (Looked at S9). Good. You have good ideas, but you need to be very clear. (Looked at the whole class). [S9] said that this doesn’t look right. They are two different things. What does he mean by two different things? Anybody wants to add to his comments? (Picked S10 to answer).

Student S10: One is time, and one is duration.

Ms. Hannah: One is time, and one is?

Student S10: Duration.

Ms. Hannah: So, this is not right. We don’t show it this way…

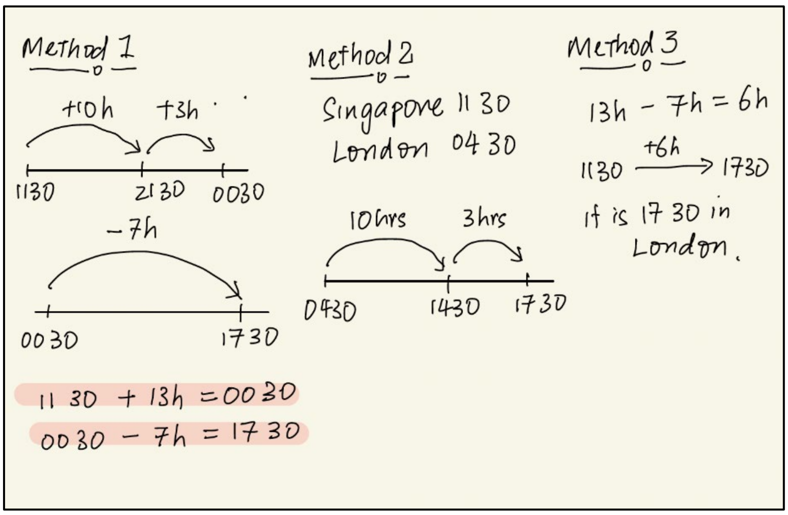

Figure 1: Three solutions presented by students (reproduced for clarity).

Ms. Hannah went on to invite another group of students to present a different solution (see Method 2 in Figure 1), which is to work in London time from the start. After eliciting these two solutions, Ms. Hannah was aware that some students had a third solution. She invited two students, S11 and S12, to present their answers (see Method 3 in Figure 1). This solution "imagined" the time in London to be 11 30 (7 hours later) and so the remaining flight time is 6 hours (13h – 7h = 6h). Hence, the required time in London will be 17 30 (6 hours after 11 30). As expected, this is a rather sophisticated reasoning for a Grade 4 student and a student, S11, had some difficulty expressing his ideas. Ms. Hannah then used this opportunity to ask the class some clarifying questions to make sure that most of the class could understand the solution before she explained the S11's solution to the class.

We see how Ms. Hannah used her students' responses to engage them in mathematical reasoning. She was deliberate in her selection of solutions. For the first group, while mostly accurate, she noticed and harnessed the affordances of students' work to highlight an important convention when working with time. For the third group, she wanted to highlight the sophistication in their reasoning. In addition, we also see how she pressed the students to clarify their understanding and engaged her students in productive revoicing. Hence, she was able to assess whether her students had actually understood the focus of the discussion (Van Zoest et al., 2023) before she explained the solutions.

DISCUSSION

Taken together, the two vignettes reaffirmed the intricacies and complexities of a mathematically productive whole-class discussion (Lobato et al., 2005; Smith & Stein, 2011; Van Zoest et al., 2023). It is not merely the use of talk moves such as collecting, clarifying, and revoicing, that make a discussion mathematically productive. As demonstrated by Ms. Hannah, teachers need to engage in both productive listening (Rinaldi, 2001) and productive telling (Lobato et al., 2005), and the talk moves must work in concert to focus the discussion on key mathematical ideas. Enabling this requires efforts to build a conducive classroom environment where it is routine for students and teachers to listen, make sense, and share their ideas. Ms. Hannah's structure for the lesson depicted in this paper was not a one-off event. In all our observations of her lessons, we saw a consistent lesson structure: Clarifying or recapping ideas→ introducing the task→ individual working on the task→ collaborating on the task→ presenting different ideas→ connecting ideas to make a mathematical point→ focused practice for mastery. This structure is similar to what researchers such as Smith and Stein (2011) had suggested. What Ms. Hannah had demonstrated was how she used this structure to create opportunities for students to share their ideas in ways that would enable her to ask focusing questions relevant to the objectives of the lesson (Herbal-Eisenmann & Breyfogle, 2005). Notwithstanding the limitations of this study, we believe that her skilful interweaving of talk moves and telling was key to unlocking the mathematical quality of her whole-class discussions.

Acknowledgement

This paper refers to data from the research project “Building a culture of collaboration and listening pedagogy in classrooms through Lesson Study for Learning Communities: An exploratory study in a primary school in Singapore” (OER19/17LKE), funded by the Office of Educational Research (OER), National Institute of Education (NIE), Nanyang Technological University, Singapore, as part of the NIE Education Research Funding Programme (ERFP). The views expressed in this paper are the authors’ and do not necessarily represent the views of NIE.

References

Brodie, K. (2010). Teacher responses to learner contributions. In K. Brodie (Ed.), *Teaching mathematical reasoning in secondary school classrooms* (pp. 139-165). Springer. https://doi.org/10.1007/978-0-387-09742-8_9

Chan, C. M. E. (2016). *Targeting Mathematics Workbook 4B*. Star Publishing Pte Ltd.

Choy, B. H. (2016). Snapshots of mathematics teacher noticing during task design. *Mathematics Education Research Journal*, *28*(3), 421-440. https://doi.org/10.1007/s13394-016-0173-3

Constance, K., & Kelly, A. R. (2012). Elapsed time: Why is it so difficult to teach? *Journal for Research in Mathematics Education*, *43*(3), 296-315. https://doi.org/10.5951/jresematheduc.43.3.0296

Herbal-Eisenmann, B. A., & Breyfogle, M. L. (2005). Questioning Our "Patterns" of Questioning. *Mathematics teaching in the middle school*, *10*(9), 484-489.

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, *36*(2), 101-136.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Harvard University Press.

Ministry of Education-Singapore. (2012). *Primary mathematics teaching and learning syllabus*. Curriculum Planning and Development Division.

O'Connor, M. C., & Michaels, S. (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. *Anthropology & Education Quarterly*, *24*(4), 318-335.

Rinaldi, C. (2001). The pedagogy of listening: The listening perspective from Reggio Emilia. *Innovations in Early Education: The International Reggio Exchange*, *8*(4), 1-4.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics Inc.

Thomas, M., Clarke, D. M., McDonough, A., & Clarkson, P. C. (2023). Assessing students’ understanding of time concepts in Years 3 and 4: insights from the development and use of a one-to-one task-based interview. *Mathematics Education Research Journal*, *35*(1), 1-22. https://doi.org/10.1007/s13394-023-00451-3

Van Zoest, L. R., Stockero, S. L., Peterson, B. E., & Leatham, K. R. (2023). (Counter)Productive Practices for Using Student Thinking. *Mathematics Teacher: Learning and Teaching PK-12*, *116*(4), 244-251. https://doi.org/10.5951/mtlt.2022.0307

ESSENTIAL PROGRAM FEATURES IDENTIFIED BY STUDENTS WORKING TOWARD A DOCTORATE IN MATHEMATICS EDUCATION

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What are the essential components of a doctorate program in mathematics education or didactics of mathematics concerning research, coursework, seminars, and collaboration? The purpose of this study was to learn from doctoral students across the world about how their programs in mathematics education are preparing them for research and teaching in mathematics education; how their programs provide academic research and writing support; and what they view as missing from their experiences. Online surveys, along with follow-up interviews from a subset of survey respondents, indicated that doctoral students from 17 different countries stressed the importance of international collaboration, examining fundamental theories of learning mathematics, and identified a need for more support with academic writing.

INTRODUCTION

Over the past two decades, the growing internationalization of institutions of higher education has been one of the most widely discussed and researched aspects of further education (e.g., Altbach & Knight, 2007; de Witt & Deca, 2020). This period has also seen a significant increase in research into the development of and issues around mathematics teacher educators (MTEs). One subset of MTEs lacking a robust research base is the group of holders and pursuers of doctorates in mathematics education or didactics of mathematics, depending on your geography and background. Existing research on mathematics education doctorates, although limited, has highlighted the great variability in doctoral preparation and programs (e.g., mathematics knowledge preparation, research training) and focused on the potential to identify a common core of knowledge and experiences that would prepare graduates for diverse careers (Goos & Beswick, 2008; Kilpatrick & Spangler, 2015; Reys, 2002).

The research presented here is part of a larger project designed to identify features of doctoral programs in mathematics education with the potential to make up a common core set of experiences, practices, and expertise. Such a core holds promise to become what Engwall (2016) identified as the "most significant mode of internationalization in higher education . . . the Import of Ideas to the home institution" (p. 222). According to Engwall (2016), the Import of Ideas takes place mainly through international contacts and collaborations, and through national faculty members' selection of course literature. Therefore, identifying features of doctoral programs in mathematics education that remain essential across institutions and

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 144–151). PME.

countries has the potential to become part of a core set of experiences, practices, and expertise for any mathematics education doctoral program, regardless of where it is located (Grevholm et al., 2008).

In this paper, we report on ongoing international research to collect and examine data about the experiences, practices, and expertise of individuals with or working toward a doctorate in mathematics education. The following questions guide the research presented here: 1) What features of doctoral programs, across countries, do individuals pursuing a doctorate in mathematics education identify as being essential? 2) Which aspects of research and teaching in mathematics education do doctoral students need more support with?

METHODS

A combination of purposive and convenience sampling was used to identify and contact (via email) potential participants for the larger study, which is composed of individuals with a doctoral degree in mathematics education (or didactics of mathematics) or currently working toward such a degree. Several proceedings from international and regional mathematics education conferences from the past five years (e.g., CERME 13, MERGA 45, NORMA 20, PME 46, PME-NA 45, The Mathematics Education for the Future Project, XVI CIAEM) were used to obtain the email addresses of potential study participants. Next, potential participants were emailed a letter introducing the study and inviting them to click on a link to a consent form and survey (the survey link is still active and available at https://tinyurl.com/DocMathEd). The survey was designed to identify the experiences, practices, and expertise of individuals holding or pursuing a doctorate in mathematics education. It was hoped that respondents would forward the email and link to their colleagues and/or doctoral students, which occurred in several instances. The last survey question asked if participants were available for a follow-up interview about their doctoral program experiences.

Participants

Survey participants for this report comprised 28 mathematics education doctoral students from 17 countries: one participant from Australia, Austria, Brazil, Columbia, France, Germany, Ghana, India, Israel, Mexico, United Kingdom, and Zambia; two participants from each of Norway and Spain; 3 participants from Indonesia; 4 participants from Malaysia; and 5 participants from the United States. Each participant self-identified as someone working toward a doctorate in mathematics education (or didactics of mathematics). Twenty-six of 28 (92.9%) participants indicated they planned to pursue a career as a university faculty member (e.g., professor, lecturer, researcher) upon graduation. Thirteen of 28 participants (46.4%) were pursuing their degree in a Department of Mathematics Education, 6 participants (21.4%) in a Department of Mathematics or Mathematical Sciences, 4 participants (14.3%) in a Department of Education, and 5 participants (17.9%) were pursuing their degree in some other department (e.g., Department of Linguistic, Scientific, and

Mathematics Education). Finally, of the 18 participants who indicated their willingness to participate in a follow-up interview, five scheduled, attended, and responded to a 30-minute, semi-structured interview.

Data Collection

Participants were asked a series of survey questions regarding how important they believed specific features (see Table 1) were to a doctoral program in mathematics education.

Doctoral Program Feature	Doctoral Program Feature
Analyze, design, and evaluate mathematics curricula	Develop broad and deep knowledge of the big ideas in pre-K–14 (e.g., ages 2-20 years) mathematics
Study the history of mathematics education	Examine how the big ideas in pre-K–14 (e.g., ages 2-20 years) mathematics develop in students
Examine historical, social, political, and economic factors that influence mathematics education	Utilize technology as a tool of inquiry in mathematics teaching and learning
Examine current and historical research in the field of mathematics education	Design learning experiences for students and teachers that utilize technology
Examine and compare fundamental theories of learning mathematics	Supervise field experiences for prospective (pre-service, student) mathematics teachers
Examine the influence of curriculum frameworks, standards, and/or competencies on school mathematics programs	Examine issues of diversity, equity, and inclusion in mathematics learning and teaching
Examine and compare different forms and purposes of assessment	

Table 1: Doctoral Program Features

Responses were limited to “Very Important,” “Moderately Important,” “Slightly Important,” “Not Important,” and “Not Necessary/Not Required.” All 28 participants responded to 13 of these Likert-type level of importance questions. The interview questions for participants pursuing a doctorate in mathematics education focused on the strengths of their program, aspects where they needed more support, prior degrees, teaching experience, and current research.

Analysis

The Likert-type level of importance questions were analyzed by weighing each possible anchor response as follows: “Very Important” = *4*, “Moderately Important” = *3*, “Slightly Important” = *2*, “Not Important” = *1*, and “Not Necessary/Not Required” = *0*. Next, the number of responses for each anchor was determined and the sum of points was calculated. For example, the question focused on the importance of the doctoral program feature “Analyze, design, and evaluate mathematics curricula” received the following responses: “Very Important” was selected by 19 participants; “Moderately Important” and “Slightly Important” were each selected by four participants; “Not Important” by no participants; and “Not Necessary/Not Required” by one participant. Therefore, the doctoral program feature “Analyze, design, and evaluate mathematics curricula” received a weighted score of $19 \times 4 + 4 \times 3 + 4 \times 2 + 0 \times 1 + 1 \times 0 = 96$. A sequence of Fisher’s exact tests for a 2×5 contingency table was also performed to determine the significance of associations between participants’ selected importance levels (e.g., “Moderately Important”) and each pair of program features (e.g., “Examine and compare fundamental theories of learning mathematics” and “Analyze, design, and evaluate mathematics curricula”).

Thematic analysis was implemented for the interview data, so the researchers could analyze emerging themes that aligned with the research questions for those aspects of essential components of doctorate programs in mathematics education from the larger study that is currently in progress (Braun & Clarke, 2013). This process was structured around both research questions in this report, to focus the findings on essential components, strengths of doctoral programs, and where more support is needed.

RESULTS

The results section is divided into two subsections, each focused on one of the research questions. The first subsection provides results from the surveys based on the first research question.

What features of doctoral programs, across countries, do individuals pursuing a doctorate in mathematics education identify as being essential?

The top five weighted scores from the series of survey questions regarding participants’ views about the importance of various doctoral program features are illustrated in Table 2. These results are ranked from the largest to smallest weighted score value.

Doctoral Program Feature	Weighted Score	Very Important
Examine and compare fundamental theories of learning mathematics	100	21
Analyse, design, and evaluate mathematics curricula	96	19

Develop broad and deep knowledge of the big ideas in grades pre-K-14 mathematics (e.g., ages 2-20 years)	93	14
Examine current and historical research in the field of mathematics education	93	13
Utilize technology as a tool of inquiry in mathematics teaching and learning	93	13

Table 2: Essential Features of Doctoral Programs in Mathematics Education

Results of the sequence of Fisher's exact tests indicated a significant association between participants' selected importance levels (e.g., "Moderately Important") and the doctoral program feature "Examine and compare fundamental theories of learning mathematics" with each feature illustrated in Table 3.

Doctoral Program Feature	p ($\alpha = 0.05$)
Study the history of mathematics education	0.0051
Examine historical, social, political, and economic factors that influence mathematics education	0.0317
Examine current and historical research in the field of mathematics education	0.0401
Utilize technology as a tool of inquiry in mathematics teaching and learning	0.0401
Design learning experiences for students and teachers that utilize technology	0.0448

Table 3: Significant Fisher's Exact Test Results

The results of Fisher's exact test ($p = 0.0336$) also indicated a significant association between participants' selected importance levels (e.g., "Very Important") and the doctoral program features "Analyze, design, and evaluate mathematics curricula" and "Study the history of mathematics education." Finally, results of Fisher's exact test indicated the associations between participants' selected importance levels for all other survey feature pairings (e.g., "Examine and compare fundamental theories of learning mathematics" and "Analyze, design, and evaluate mathematics curricula") were not significant. Therefore, being provided with opportunities to examine and compare fundamental theories of learning mathematics was an essential feature across programs and countries. Furthermore, the remaining four features identified in Table 2 have the potential to serve as essential common experiences.

The next subsection addresses findings from both the open-ended survey items and interviews and addresses the second research question. Participants' open-ended item responses focused on essential components of mathematics education doctoral programs. Finally, participants' interview responses focused on the strengths of the doctoral programs from the perspectives of current students, what was lacking, in their programs, and whether participants believed there were core components that every mathematics education doctorate program should include.

Which aspects of research and teaching in mathematics education do doctoral students need more support with?

Participants discussed the importance of international conference experiences and collaborations and indicated such components should be a required part of a core set of experiences in any doctoral program. According to one participant, "Exposing doctoral students to international forums, webinars, ways to communicate with other countries, this can't be done only by reading. Knowledge can be gained by lectures, readings, but . . . international collaboration is needed." This participant is required to present at least at one international conference as part of their doctoral program, a requirement for which they are grateful and believe should be part of every program.

Academic writing support was most often identified as lacking in mathematics education doctorate programs. Participants felt they were expected to have acquired these skills before entering the doctoral program. Unfortunately, varying backgrounds in prior degree programs did not always prepare these doctoral students for academic writing in their research. For example, students with prior degrees in pure mathematics have experience writing proofs but not necessarily developing literature reviews. Other participants expressed a feeling of isolation in their research endeavors and a desire for additional opportunities to collaborate with mathematics education colleagues and peers. One participant indicated that although their program provided academic writing support, they felt isolated because they were attending courses and seminars online as their residence was in a different country from their home institute.

In addition to the lack of support for academic writing, another participant emphasized the lack of consistency of support from graduate supervisors. This participant stated, "Mentorship and advising are vital – [but] different from person to person; Some PhD students get completely isolated by their advisor, while others get brought into research projects and spend time writing with their advisor." This doctoral student requested that rules and regulations for the supervising process be implemented so that every student has a productive and beneficial experience. Despite this lack of consistency, this participant provided positive feedback regarding the core knowledge they acquired in the field of mathematics education during coursework.

According to interview participants, core components of a doctoral program in mathematics education, in addition to international conference opportunities, should include "emphasizing a scientific mentality and mentoring in a culture of research." Furthermore, participants stated that collaborative research and being treated as part of a research team not only promotes collegiality but also provides a greater contribution to the field of mathematics education. Two participants emphasized the importance of teaching experiences, understanding deductive skills and students' thinking processes, and interacting with students as they engage in mathematics. This combination of perspectives on the importance of both teaching and research components can help to provide a guideline for essential components of any doctoral program in mathematics education.

Responses from the open-ended questions in the survey aligned with those from the interview, which focused on how teaching experiences benefitted them as a researcher in didactics of mathematics. According to one participant, “For me, it is very important, the interaction between theory and practice, and the constant involvement with school reality; for example, through the supervision of educational internship experiences of prospective teachers.” This should be a call to action for any mathematics education professor with extensive experience in higher education—to stay involved and relevant with current teaching practices and the reality of current classroom experiences.

Participants’ survey responses also spoke to the importance of international collaboration. One respondent stated, “I think that understanding the reality of other countries and seeking to learn new strategies and learn about other points of view are important to boost the quality of mathematics teaching at all levels in our country.” Although the desire to focus on the needs of the student body a teacher or professor is currently working with is understandable, there is so much that can be learned from connecting with an international network of mathematics educators through a variety of forums.

DISCUSSION

Doctoral students in mathematics education can pursue many different career paths after graduating from their program, yet there are essential features most participants in this study indicated were necessary for their future success. As Herbst (2023) suggests in his recent editorial, we all serve to contribute to the field of knowledge in mathematics education and this process should be more collaborative and supportive. One way to achieve more collaboration and support is to incorporate additional modes of internationalization into further education. Of course, mathematics education university professors and researchers should maintain their autonomy in teaching and lines of inquiry; still, some common ground among programs, expectations, expertise, and support for doctoral students could lead to greater contributions to our field.

More attention and support in obtaining funding would be beneficial to current doctoral students. This suggestion was requested not only by study participants but also by experts in didactics of mathematics during the validation stage of this study’s instruments, who lamented about not learning about the funding process earlier in their careers. Finally, continued professional development opportunities related to emerging digital technologies was requested by study participants and should be part of any doctoral program, regardless of the country, institution, or grade-level focus.

This report is part of a larger, international study with the intent to continue the discussion and promote actions toward more cohesive expectations, practices, and expertise for doctoral programs in mathematics education. Such discussions and actions have the potential to develop guidelines for robust mathematics education doctoral programs. For instance, one participant identified the need for a study such as the one presented here, especially in light of the emerging number of doctoral programs in didactics of mathematics in Indonesia. The importance of international collaboration cannot be understated, and essential components of teaching and research in mathematics education should continue to be identified and encouraged.

References

Altbach, P. G., & Knight, J. (2007). The internationalization of higher education: Motivations and realities. *Journal of Studies in International Education*, *11*(3-4), 290-305.

Braun, V., & Clarke, V. (2013) *Successful qualitative research: A practical guide for beginners.* SAGE Publication.

de Wit, H., & Deca, L. (2020). Internationalization of higher education, challenges and opportunities for the next decade. In S. Bergan, D. Crosier, M. Gaebel, C. M. Haj, R. Iucu, L. Matei, D. Orr, H. de Wit, & T. Zhang (Eds.), *European higher education area: Challenges for a new decade* (pp. 3-11). Springer.

Engwall, L. (2016) The internationalisation of higher education. *European Review*, *24*, 2, 221-231.

Goos, M., & Beswick, K. (Eds.). (2021). *The learning and development of mathematics teacher educators: International perspectives and challenges*. Springer.

Grevholm, B., Sullivan, P., & Reys, R. (2008). *Discussion Group (DG) 12: Rethinking doctoral programmes in mathematics education*. University of Latvia.

Herbst, P. (2023). The doctorate is part of the infrastructure of our research field. *Journal for Research in Mathematics Education*, *54*(5), 288-294.

Kilpatrick, J., & Spangler, D. A. (2015). Educating future mathematics education professors. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed.) (pp. 297-310). Routledge.

Reys, R. E. (2002). *Principles to guide the design and implementation of doctoral programs in mathematics education: A task force report for the association of mathematics teacher educators* (ERIC Document Reproduction Service No. ED 477 310). Association of Mathematics Teacher Educators.

AN EMPIRICAL EVALUATION OF USING INSTRUCTIONAL VIDEOS IN DIFFERENTIATED INSTRUCTION FOR EIGHTH GRADERS' LEARNING OF MATHEMATICAL PROBLEM SOLVING

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This study aimed to develop instructional videos for assisting differentiated eighth graders' learning in mathematical problem solving. To evaluate the effect of the intervention strategy, we used pre- and posttests on the participants' performance and learning motivation as well as semi-structured interviews to understand high, medium, and low-achieving participants' perceptions of the learning experience. The results showed that the teaching designs had significant effects on improving students' performance and learning motivation in mathematical problem solving. The interviews revealed that the effect may have resulted from using the instructional videos for individual learning pace, clarifying the concepts applied for problem solving, and providing clear instructional guidance, especially for low-achieving students.

INTRODUCTION

Mathematical problem solving, including the processes of formulating, employing, and interpreting, is an important dimension in the Programme for International Student Assessment (PISA) framework for mathematical literacy assessment (OECD, 2022). Multiple studies have indicated that improving students' mathematical problem-solving ability is crucial for cultivating mathematical literacy (Lesh, 2007). In particular, when solving PISA-like problems, students with various levels of mathematical literacy may learn at different paces and encounter distinct challenges. Differentiated instruction has been justified as an effective method to provide appropriate learning opportunities for a variety of students. With the development of technology, instructional videos come in handy to address the issues of differentiated instruction.

As technology has developed, instructional videos have become an important element in mathematics education. They not only provide visual learning materials but also meet the learning needs of different students through differentiated instruction (Kay et al., 2012). Based on Mayer's multimedia learning theory and Sweller's cognitive load theory, current empirical research has found that interactive multimedia instructional design significantly improves students' reasoning abilities (Amir et al., 2018). Khan-style instructional videos, which demonstrate mathematical equations through freehand writing instead of computer-rendered fonts, effectively support students' understanding and learning processes by combining visual elements and narrative content (Hew, 2018). Additionally, multimedia elements such as videos and animations have a positive impact on learning (GebreYohannes et al., 2016). However, there is a lack of relevant research on using instructional videos to enhance students'

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 152–159). PME.

mathematical problem-solving ability and learning motivation. Recognizing the necessity of differentiated instruction and the importance of cultivating mathematical problem-solving ability, this study explored the development of instructional videos for learning how to solve PISA-like problems, and evaluated their impact on students' problem-solving performance and learning motivation.

The research questions are as follows:

- Does the intervention strategy enhance the learning motivation of eighth-grade students?
- Does the intervention strategy improve the problem-solving performance of eighth-grade students?
- What are the learning experiences of eighth-grade students with high, medium, and low academic performance regarding this instructional intervention?

METHOD

Design and Implementation of Instructional Videos and Worksheets

The instructional videos and worksheets were primarily derived from open-ended questions in the Taiwan junior high school major examinations. The design of the instructional videos applied the media effect of cognitive load theory, utilizing voiceover to convey information instead of relying solely on text, effectively reducing learners' cognitive burden. (Mousavi et al, 1995). The synchrony principle emphasizes presenting visual and auditory information simultaneously; for example, synchronizing voiceover with mathematical equations can significantly improve learning effectiveness (Mayer et al, 2016). In addition, Hew (2018) analyzed six styles of instructional video and found that Khan-style videos featuring instructor-led demonstrations using a handwritten method had the highest recall and application test performance.

The study used a single-group pretest-posttest design, with a 4-week teaching intervention consisting of one 45-minute class per week. Students were divided into high, medium, and low learning performance groups, with six, eight, and six students in each group, where the teacher provided adaptive support according to the group's need. The instructional videos were pre-recorded by the researcher and presented the problem-solving process using Khan-style explanations along with the teacher's dynamic headshot, with each worksheet question accompanied by a QR code.

This study implemented a cyclical teaching model to enhance students' understanding of mathematical literacy questions before proceeding to problem solving. Each teaching cycle consisted of four stages. First, students independently considered the questions to foster their initial understanding. Second, they engaged in homogeneous subgroup discussion, which may benefit peer interaction and understanding of the questions supported by the teacher. Next, students watched the instructional video individually. Finally, the teacher took over the last 3 minutes to guide students by summarizing and reflecting on problem-solving strategies. This last stage allowed students who had not watched the whole video to be aware of the core problem-solving strategy. Each class could go through the teaching cycle three times.

RESEARCH INSTRUMENT

Problem-Solving Performance

To measure the difference in students' mathematical literacy problem-solving performance before and after the intervention, this study selected open-ended problems from Taiwan junior high school major examinations as the test questions. These questions were chosen to align with the knowledge typically acquired by eighth-grade students, focusing on numerical and quantitative concepts, which are central to this intervention study. Considering the teaching time, we selected six major questions, each containing two sub-questions, resulting in a total of 12 sub-questions, and classified them into easy, medium and hard levels.

Based on the assessment framework pyramid analysis, we divided the test questions into three major sections for the pretest and posttest. It was confirmed in the pilot test that there was no significant difference in difficulty between the pretest and posttest questions. The sample was selected based on convenience sampling, selecting students from one class and dividing them into two groups. One group of students took the pretest, while the other took the posttest. Non-parametric related sample tests were conducted on the scores of each question according to the scoring criteria to evaluate the consistency between the pretest and posttest questions. We calculated the W and p values for each corresponding question using the Wilcoxon signed-rank test. The results showed that the W values for the three corresponding questions were 0.447, 0.577, and 1, with corresponding p values of 0.655, 0.564, and 0.317. For the total score, the W value was 1.342, and the p-value was 0.18. These results indicate no significant difference between the pretest and posttest questions, confirming that the difficulty of the pretest and posttest questions was comparable and that they could be used to assess the differences in student performance before and after learning.

Learning Motivation

The motivation for learning was measured using a questionnaire developed by Cherng and Lin (2001), which was adapted from the "Motivated Strategies for Learning Questionnaire" (MSLQ) (Pintrich et al., 1991). The original questionnaire was designed for university courses, while the version by Cherng and Lin (2001) was revised for middle school students. It comprises two components: the value component reflects the importance of learning activities to students and their interests, while expectancy is related to learners' self-efficacy and beliefs. Based on the results of testing 4,082 middle school students, the internal consistency reliability (Cronbach's α) ranged from .55 to .87, and the test-retest reliability ranged from .57 to .87. The questionnaire used a Likert 4-point scale, with 4 indicating *strongly agree*, 3 indicating *agree*, 2 indicating *disagree*, and 1 indicating *strongly disagree*.

Semi-structured interviews on learning experiences

This study employed semi-structured interviews to gain insights into students' learning experiences with instructional videos. The interview questions were designed to include fixed questions to ensure data comparability and open-ended questions to encourage students to express their deeper feelings and thoughts. Additionally, the follow-up questions were adjusted in real time based on students' responses to deepen the understanding of their learning process. Table 1 below presents the detailed interview questions for the interview purpose, fixed and open-ended questions.

Furthermore, this study focused on analyzing feedback from students with different learning achievements (high, medium, and low) to compare the impact of instructional interventions on their learning experiences. Post-class interviews spanning 3 weeks collected students' overall perceptions of the instructional intervention, encompassing both positive and negative feedback. The interview data underwent systematic processing and analysis using thematic analysis.

Interview purpose	Fixed Questions	Open-ended Questions
Student's learning experience	In what specific ways will you benefit from the instructional videos?	Can you provide an example of a time when you encountered difficulties while watching instructional videos, and how you resolved them?
General Feelings about Teaching	What are your thoughts about integrating instructional videos into mathematical literacy learning? Do you have any suggestions for instructional videos and teaching?	Can you provide an example to illustrate the positive and negative learning experiences brought by instructional videos?

Table 1. The detailed interview questions for the interview purpose

Data Collections and Analysis

This study adopted mixed research methods. Quantitative data included mathematical literacy problem-solving performance, based on the PISA mathematical literacy assessment standards. A single-group pretest-posttest approach was adopted, and paired sample *t* tests were conducted to compare the average scores before and after the test, as well as learning motivation, and the revised version of the Motivated Strategies for Learning Questionnaire (MSLQ) was administered. Qualitative data were analyzed using thematic analysis, focusing on students' learning experiences. Semi-structured interviews were conducted to gain an in-depth understanding of students' emotional responses to the instructional videos and their impact on learning.

RESEARCH RESULTS

Problem-Solving Performance

The pretest and posttest consisted of three major questions, and were scored according to the grading standards published by the National Examination Center of Taiwan. A score of 3 indicated appropriate strategy, reasonable and complete expression; 2 indicated a generally complete strategy with calculation errors or lack of demonstration of the rationality of some steps; 1 indicated insufficiency to solve the problem, or failure to fully transform the problem into a mathematical question; while 0 indicated a vague strategy, blank problem-solving process, or an answer that was irrelevant to the question. Therefore, the total score for the three questions ranged from 0 to 9. Research results based on the mathematical literacy problem-solving performance of participating students showed a significant increase in average scores in the posttest compared to the pretest (pretest average score of 2.05, posttest average score of 5.75), $t(19) = -8.865$, $p = .001$. This result indicates that instructional intervention has a significant positive impact on student learning outcomes.

Furthermore, the improvement in students' pretest and posttest problem-solving performance can be seen from the scores(see Table 2). The scores were divided into a low score group with 0-3 points, a medium score group with 4-5 points, and a high score group with 6-9 points. The distribution of the pretest and posttest scores is presented in a table. The data showed an upward triangular distribution, indicating a positive effect on students with different learning performances. In particular, among the low score group students, seven improved to the high score group, three improved to the medium score group, and four students in the low score group also showed improvement. Students with pretest scores in the medium score groups also improved to the higher score range. This result may be attributed to the intervention of instructional videos which enhanced the understanding of basic concepts and skills, and the in-depth learning of literacy test questions for the first time, resulting in improved learning outcomes after 4 weeks.

Pretest/Posttest Score Range	Posttest 0-3	Posttest 4-5	Posttest 6-9
Pretest 0-3	4	3	7
Pretest 4-5	0	1	4
Pretest 6-9	0	0	1

Table 2. Pretest and posttest improvement score range

Learning Motivation

The test results showed an improvement in students' problem-solving performance. As for learning motivation, the researchers had already announced to the students during the informed consent stage that instructional videos would be integrated into the math classes, and the students expressed anticipation and excitement. Therefore, an analysis was conducted on the two major aspects of value and expectation in learning motivation before and after the intervention. The paired sample *t* test conducted using SPSS showed that the average score of students' learning motivation increased from 2.8655 in the pretest to 3.4775 in the posttest, with a *p*-value of .044. This result indicates the effectiveness of the instructional intervention in terms of enhancing students' learning motivation. It reflects students’ positive response to interactive learning methods, as the videos enhanced their understanding of mathematical concepts and their perceptions of intrinsic value.

Learning Perceptions

The researchers conducted semi-structured interviews with students of different learning performances after class every week and used thematic analysis to identify the key aspects of students' learning experiences. During the coding process, the researchers summarized 15 relevant themes, but they only presented the perspectives mentioned by high, medium, and low-performing students.

After categorization and analysis, students' feedback concentrated on the perceived ease of use and usefulness. Perceived ease of use reflected students' evaluation of the convenience of implementing teaching interventions. High-performing students mentioned the flexibility of watching videos and the ability to watch videos to address their own questions freely. Medium-performing students mentioned the need to watch videos repeatedly to clarify their questions, while low-performing students emphasized the necessity of extra time for understanding. Perceived usefulness reflected students' assessment of how teaching interventions improved learning outcomes. High-performing students considered that instructional videos enhanced their problem-solving skills and ability to express their ideas clearly. Medium-performing students felt that the videos helped them reflect on and identify learning blind spots, while low-performing students believe that the instructional videos helped them understand the questions and made them more willing to learn.

In the final summary interview and feedback after the last class, high-performing students felt that the videos provided a high degree of autonomy, thereby enhancing their problem-solving abilities and clarity of expression. Medium-performing students emphasized the importance of combining instructional videos with traditional teacher explanations to clarify misunderstood concepts, believing that this combination effectively helped them reflect and identify learning blind spots. Low-performing students particularly valued the benefits of videos in providing learning flexibility and clear guidance, which prompted them to learn and understand the mathematical problems more actively. In summary, this study confirmed the effectiveness of instructional videos in enhancing students' learning motivation, improving problem-solving abilities, and meeting different learning needs, providing important empirical support for future instructional design.

DISCUSSION AND CONCLUSION

This study confirms that the instructional process design of differentiated instruction significantly enhances students' mathematical literacy, problem-solving abilities, and learning motivation. Reflecting on the implementation effects of instructional video design can reveal its potential effectiveness. For example, when students watch instructional videos, they may become more engaged in learning because they can first understand the essence of the problem, and through group discussions, they promote collaborative learning, enhancing a deeper understanding of the problem. Therefore, we suggest that future research can refer to this instructional process design, and in terms of learning experience, students' positive feedback on perceived ease of use and usefulness highlights the role of instructional videos in promoting students' autonomy and understanding of different learning performances. High-performing students praised the autonomy of the videos; medium-performing students valued the integration of videos with traditional teaching; and low-performing students affirmed the learning support provided by the videos.

In summary, this study confirms the effectiveness of instructional videos in mathematics education, as they significantly enhance students' mathematical literacy, problem-solving performance, learning motivation, and positive learning experiences. Students reported that the videos enhanced their autonomous learning and deepened their understanding of problems. Future research should refine instructional content for students with different learning performances and explore the integration of more diverse teaching methods and learning resources. The limitations of this study include the need for a more detailed exploration of the impact of differentiated design on students of different grades and the need to combine practical operations with instructional videos for teaching objectives focused on concept development. Therefore, future research should expand the sample size, consider educational diversity, and enhance the generalizability of the research results.

References

Amir, M. F., Hasanah, F. N., & Musthofa, H. (2018). Interactive multimedia based mathematics problem solving to develop students' reasoning. *Int. J. Eng. Technol*, 7(2.14), 272-276.

Cherng, B.-L., & Lin, C.-S. (2001). The construction and validation of a self-regulated learning scale for secondary students. *Journal of Testing, 48(1)*, 1-41.

GebreYohannes, H. M., Bhatti, A. H., & Hasan, R. (2016). Impact of multimedia in Teaching Mathematics. *International Journal of Mathematics Trends and Technology, 39(1)*, 80-83.

Hew, K. F., & Lo, C. K. (2020). Comparing video styles and study strategies during video-recorded lectures: Effects on secondary school mathematics students' preference and learning. *Interactive Learning Environments, 28(7)*, 847-864.

Kay, R., & Kletskin, I. (2012). Evaluating the use of problem-based video podcasts to teach mathematics in higher education. *Computers & Education, 59(2)*, 619-627.

Lesh, R. (2007). Problem solving and modeling. *Second handbook of research on teaching and learning, 2*, 763-804.

Mousavi, S. Y., Low, R., & Sweller, J. (1995). Reducing cognitive load by mixing auditory and visual presentation modes. Journal of educational psychology, 87(2), 319.

Mayer, R. E., & Moreno, R. (2016). Nine ways to reduce cognitive load in multimedia learning. In *Cognitive Load Theory* (pp. 43-52). Routledge.

OECD (2023), "PISA 2022 Mathematics Framework", in PISA 2022 Assessment and Analytical Framework, OECD Publishing, Paris, https://doi.org/10.1787/7ea9ee19-en.

Pintrich, P. R. (1991). A manual for the use of the Motivated Strategies for Learning Questionnaire (MSLQ).

EFFECTS OF THE TYPE OF ARGUMENT ON STUDENTS' PERFORMANCE IN PROOF-RELATED ACTIVITIES

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Different types of arguments, such as empirical arguments and generic proofs have been discussed in the literature regarding students' convictions and their potential for proof comprehension. However, their influence on proof-related activities is still not clear. The experimental study presented in this paper aims at closing that gap. Data from N=430 first-year university students suggests that generic proofs are easier to understand than ordinary proofs. Moreover, it indicates that students' self-reported conviction by different types of arguments does not reflect their actual conviction of the truth of statements. The findings highlight students' difficulties with the relation between the validity of a statement and that of its proof and provide a basis for developing courses in a manner that eases the transition to proof-based mathematics.

INTRODUCTION

Proofs are undoubtedly fundamental for Mathematics. In consequence, understanding proof has been one focus of mathematics education research for many decades and proof and argumentation are central goals in national educational standards worldwide (e.g., Kultusministerkonferenz, 2012; National Council of Teachers of Mathematics, 2000). Students at different school levels nevertheless struggle with proof and argumentation (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000). The lack of proof skills is particularly relevant for students entering university, because in many countries this coincides with the transition to proof-based mathematics. Students' insufficient proof skills and understanding are in fact often identified as main reasons for students' difficulties with mathematics at the transition from school to university (e.g., Gueudet, 2008).

To make proofs more accessible to students, different types of arguments have been proposed as educational tools. However, little is known regarding the influence of the type of argument on students' proof comprehension and other activities (e.g., Mejía Ramos et al., 2012). This paper therefore aims to investigate the effects of reading different types of arguments (no arguments, empirical arguments, generic proofs, and ordinary proofs) on first-year students' performance in several proof-related activities.

PROOFS AND OTHER TYPES OF ARGUMENTS

While proof plays a central role in mathematical practice, no precise definition for it exists. An often cited characterization is given by Stylianides (2007), who views proof as a mathematical argument containing (a) a set of accepted statements (e.g., definitions, axioms, theorems, …), (b) valid and known forms of reasoning (e.g.,

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application of logical rules of inferences, use of definitions, construction of counterexamples, …), and (c) appropriate and known forms of expression (e.g., linguistic, physical, pictorial, symbolic, …). This characterization of proof highlights the importance of context and the individuals who construct and evaluate the proofs. Because *ordinary proofs* (those that are typically constructed by mathematicians) are often not accessible to students, for instance, because of involved symbolic forms of expression and the level of abstraction, other types of arguments have been introduced and discussed in the literature regarding their potential for the learning of proof and argumentation. One prominent example is the *generic proof*. These types of arguments are based on specific observations, which reveal a structure that can be generalized to hold for a whole class of objects (e.g., Rowland, 2001). Researchers assume that "a generic proof makes the chain of reasoning accessible to students by reducing its level of abstraction" (Dreyfus, Nardi, & Leikin, 2012, p. 204). The usage and role of *empirical arguments*, i.e., arguments that are based on verifications of a (small) number of cases, is more controversial. On the one hand, it is argued that empirical arguments are essential for problem exploration and for gaining an intuitive understanding of the statement and its validity within an axiomatic system (e.g., de Villiers, 2010), and that mathematicians also make use of these types of arguments (e.g., Weber, 2013). On the other hand, researchers assume that students' usage of and conviction by empirical arguments may indicate an insufficient understanding of proof and its generality (e.g., Conner, 2022). The level of conviction and the awareness of the limitations of empirical arguments may thus be essential.

PROOF-RELATED ACTIVITIES AND THE INFLUENCE OF DIFFERENT TYPES OF ARGUMENTS

The main proof-related activities that are usually being distinguished in research on proof and argumentation are *proof reading*, *proof construction*, and *proof presentation* (e.g., Mejía Ramos & Inglis, 2009). This paper focuses on proof reading, an activity which is under-represented in prior research (Mejía Ramos & Inglis, 2009). It contains two central sub-activities: *proof comprehension* and *proof evaluation*.

Proof comprehension is seen as essential for the learning of mathematics at university level as students are frequently confronted with proofs in lectures and textbooks (Mejía Ramos et al., 2012). Nevertheless, only few studies have investigated how well students understand proofs (Mejía Ramos & Inglis, 2009). Even though researchers have argued that generic proofs can improve students' proof comprehension by making the ideas more accessible to them (e.g., Dreyfus, Nardi, & Leikin, 2012), respective findings are not consistent so far. A positive influence of reading generic proofs compared to ordinary proofs on first-year engineering students' proof comprehension was reported by Malek & Movshovitz-Hadar (2011), but with a small sample of only 10 students. To provide more evidence for the effect of generic proofs on proof comprehension, Lew, Weber, and Mejía-Ramos (2020) conducted an experimental quantitative study in which 106 mathematics students participated. Students were randomly assigned to either receive a generic or an ordinary proof. All participants had to complete a proof comprehension test based on the assessment model of Mejía Ramos et al. (2012). The authors did not find evidence that the generic proof lead to better proof comprehension than the ordinary proof. Even *if* generic proofs do not improve proof comprehension for mathematics university students—for which further evidence is needed—they could still potentially improve proof comprehension of high school students or students at the transition from school to university.

Comparatively more research exists on students' proof evaluation, i.e., on students' judgements of arguments regarding different aspects, such as the validity of the argument or their *conviction*. Regarding the latter (which I focus on in this paper), it is not always clear what "being convinced" refers to: a person's conviction of the validity of the argument/proof, that the argument/proof convinces the person of the validity of the statement, or something else. These differences may partly explain why findings on students' conviction of/by different types of arguments are ambiguous. Most students (and teachers) claim to be convinced by ordinary proofs—even when the proof is incorrect (e.g., Knuth, 2002; Martin & Harel, 1989). However, Weber (2010) observed that some university students in his study did not find ordinary proofs convincing, even though they accepted them as proof, which is in line with findings reported by Fischbein (1982). The degree to which students are convinced by empirical arguments and generic proofs is even less clear. Several studies have reported that many students (and teachers) claim to be convinced by empirical arguments (e.g., Knuth, 2002; Martin & Harel, 1989), but more advanced students are seemingly not (e.g., Weber, 2010). Similarly, some studies found that students claim to be convinced by generic proofs (e.g., Weber, 2010; Ko & Knuth, 2013) while others found the opposite (e.g., Lesseig et al., 2019), in particular, when the level of conviction was compared to ordinary proofs (Kempen, 2018). More research is needed to investigate how the type of argument influences students' conviction of the validity of statements and to what degree.

The *reading of statements* has not been explicitly considered as a proof-related activity, even though it seems to be highly relevant for the performance in other activities. Like proof reading, reading a statement can have different goals, for instance, the comprehension of the statement or estimating its truth/validity. With relation to proof, understanding the *generality of the statement* (as part of the comprehension of statements) seems to be of particular importance. Understanding the generality of a statement means to understand that no counterexample to a true universal statement exists. It can be seen as essential for the comprehension of mathematical statements and students' understanding of proof because it is the generality that is the defining element of mathematical proof (Heintz, 2000). To my knowledge, no findings have been reported on the influence of the type of argument on students' estimation of truth of the statement. Regarding students' and teachers' understanding of the generality, it has been found that (understanding) an ordinary proof is not sufficient for some students and teachers to be convinced that no counterexample can exist (e.g., Chazan, 1993; Knuth 2002). If and how the understanding of generality of statements differs by reading different types of arguments has not been investigated so far.

GOALS AND THE CURRENT STUDY

The prior research outlined above highlights the need for more systematical studies that investigate the effect of reading different types of arguments on students' performance in proof-related activities. This is particularly relevant for the learning of proof at the transition to proof-based mathematics, i.e., at the transition from school to university. The present study, which is part of my completed dissertation project (Damrau, in press), thus addresses the following research questions: (RQ1) How does the type of argument influence first-year university students' *self-reported conviction by the arguments* and how does this relate to their *actual conviction of the truth of the statements*? (RQ2) How does students' (self-reported) *proof comprehension* differ between students who receive generic proofs and those who receive ordinary proofs? (RQ3) How does the reading of different types of arguments influence students' *understanding of the generality of mathematical statements*?

Methods

To analyze the effect of the type of argument, I designed an experiment which was conducted at the beginning of two mathematics lectures at a large German university. In total, 430 students completed the questionnaire (67.4 % female, 31.2 % male, and 1.4 % chose not to answer). Most participants were preservice primary school teachers without mathematics as major (67.44 %), followed by mathematics students (16.28 %), preservice lower secondary school teachers (6.05 %), preservice higher secondary school teachers (5.81%), and preservice primary school teachers with mathematics as major (4.42 %). The median age of the participants was 20 years ($IQR = 21 - 19$) and about 96 % were German native speakers.

In the first part of the experiment, all participants read five universal statements (two of them known from school in Geometry, two of them unfamiliar and from Arithmetic, and one false Arithmetic statement) and respective arguments (non-general *pseudo proofs* were constructed for the false statement). The type of argument was assigned randomly. Group A (116 students) read no arguments at all, group B (112) only read empirical arguments, group C (107) generic proofs, and group D (95) ordinary proofs. After reading the arguments, all participants then had to estimate the *truth value* (absolutely/relatively sure about truth/falsity or "I have no idea") of each statement and decide whether *counterexamples* may exist (absolutely/relatively sure those exist or not or "I have no idea"). The participants in the groups A, B, and C were asked to evaluate the provided arguments regarding their *conviction* ("Does the justification convince you of the correctness of the claim?" completely, partially, not at all) and their *comprehension* ("Do you understand the justification?" completely, partially, not at all). In the second part, all participants had to complete a Cognitive Reflection Test (based on Frederick, 2005), which was used to control inter-individual differences in mathematical performance (for details, see Damrau, in press). The third and last part included demographic questions, including participants' final high school grade in mathematics. Students' correct *understanding of the generality* of each statement was defined as consistent responses (yes/no) regarding students' estimation of truth and the existence of counterexamples. If the students responded with "I have no idea" to both questions, the data for understanding generality was considered missing.

To estimate the effect of the type of argument on students' understanding of generality, generalized linear mixed models (GLMM) were calculated, because understanding generality was defined as a binary variable. Similarly, to analyze the effect of the type of argument on students' performance in estimating the truth value of statements, students' conviction, and students' proof comprehension, respectively, cumulative link mixed models (CLMM) were fitted, because the respective dependent variables are ordinal. In both cases (GLMM and CLMM), logistic link functions were used. Inter-individual differences between participants over all five repeated measurements were controlled with a random effect. Further, students' CRT score, final grade in mathematics during school, their participation in an *honors course* (in German *Leistungskurs*) in school, and their participation in a *transition course* (in German *Vorkurs*) prior to the beginning of the semester were used as control variables. The effect of the *type of statement* (true vs false, familiar vs unfamiliar) was also analyzed, however, respective findings are not presented here due to the focus of the paper (for details, see Damrau, in press). Holm's correction (Holm, 1979) was used to adjust the p-values according to the number of respective tests.

Results

The type of argument significantly influenced students' estimation of truth, as participants who received empirical arguments were more likely to correctly estimate the truth value of the statements than participants who got no arguments ($B = .44, p = .004$). Reading generic proofs had a similar but smaller effect ($B = .27, p = .088$), which did not reach significance after Holm's correction. Reading ordinary proofs had no significant effect on students' estimation of truth ($B = -.09, p = .537$). Further, participants who received generic or ordinary proofs were more likely to claim being convinced by these arguments than participants who received empirical arguments ($B = 1.7, p < .001$ *and* $B = 2.2, p < .001$, respectively). Students' self-reported proof comprehension differed as well between ordinary and generic proofs. Participants who received ordinary proofs reported lower comprehension than participants, who received generic proofs ($B = -.59, p < .001$). Reading different types of arguments seems to not largely influence students' understanding of the generality of statements. Overall, participants who received ordinary (and with a smaller effect generic) proofs were less likely to have a correct understanding than students who got no arguments ($B = -.41, p = .075$ *and* $B = -.32, p = .148$, respectively). However, these effects did not reach significance after Holm's correction. Reading *any* type of argument (in contrast to not receiving arguments) decreased the likelihood to answer "I have no idea" regarding the estimation of truth and the existence of counterexamples.

DISCUSSION

The goal of the present paper was to investigate how the reading of different types of arguments affects students' self-reported conviction by the arguments and their conviction of the validity of the statement, their self-reported proof comprehension, and their understanding of the generality of statements. Previous research provided ambiguous results regarding differences in students' comprehension of generic and ordinary proof. The findings reported by Lew et al. (2020) based on an experimental study suggest that there are no significant differences. In the present study, participants showed higher levels of proof comprehension regarding generic proofs than regarding ordinary proofs. However, in contrast to Lew et al., the present study relied on students' self-report on their proof comprehension. Thus, the relation between self-reported, *perceived* comprehension of proofs and students' actual proof comprehension should be investigated in future (experimental) studies.

Through the experimental design of the study, it became apparent that students' conviction of the validity of statements was influenced differently by reading different types of arguments than their self-reported conviction by the arguments: Students were the most convinced of the validity of the statements by reading empirical arguments, followed by generic proofs. Reading ordinary proofs had no significant effect on students' estimation of truth. In contrast, students *claimed* to be more convinced by generic and, even more, ordinary proofs than by empirical arguments, as prior research has suggested (e.g., Kempen, 2018; Ko & Knuth, 2013; Weber, 2010). Thereby, it can be assumed that their judgement was influenced by the perceived mathematical appearance/formality of the arguments, which is in line with prior findings (e.g., Healy & Hoyles, 2000; Sommerhoff & Ufer, 2019). My findings indicate that self-reported conviction of the validity of statements by arguments does not necessarily reflect how different types of arguments *actually* convince students of the validity of statements. Students seem to have an *empirical understanding* of the validity of statements and the relation to the validity of proof may not be sufficiently clear to them.

The reading of different types of arguments did not significantly influence students' understanding of the generality of statements. Only a negative effect of reading ordinary proofs was found (even though not significant after Holm's correction), which seems be related to the finding that more students *claimed* to be convinced by ordinary proofs, without this being reflected in increased conviction in the validity of the statement and the (non-)existence of counterexamples. The reading of *any* type of argument influenced students' responding behavior in that they were less likely to answer "I have no idea". Thus, reading an argument may have positively affected students' self-efficacy in that they at least had the feeling of knowing enough to decide on an answer, even though it did not lead to responding more successfully.

The reported findings can be used to develop future university courses in a manner that eases and promotes the transition to proof-based mathematics. In particular, the relation between the validity of the statement and the validity of its proof should be explicitly discussed. Empirical arguments and potentially generic proofs can support students' conviction of the validity of statements, but their limitations and the necessity of proof should be made clear. There are still very few experimental studies in mathematics education and particularly in research on proof and argumentation. To better understand relations between proof-related activities, students' understanding of proofs, and the effects of reading different types of arguments, more experimental studies are necessary. In this regard, more respective research on students' self-reports and their actual comprehension of and conviction by different types of proofs would be very valuable. Finally, future (intervention) studies may want to consider giving instructions before the reading of arguments or scaffolding during the reading of arguments to support students' comprehension of (the validity of) arguments and the related understanding of the validity of the statements.

References

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24(4), 359–387.

Conner, K. A. (2022). A multi-faceted framework for identifying students' understanding of the generality requirement of proof. *International Electronic Journal of Mathematics Education*, 17(4), em0702. https://doi.org/10.29333/iejme/12270

Damrau, M. (in press). Understanding the generality of universal statements. An experimental study at the transition from school to university. Springer.

de Villiers, M. (2010). Experimentation and proof in mathematics. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives* (pp. 205–221). Springer. https://doi.org/10.1007/ 978-1-4419-0576-5_14

Dreyfus, T., Nardi, E., & Leikin, R. (2012). Forms of proof and proving in the classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education. The 19th ICMI study* (pp. 191–214). Springer. https://doi.org/10.1007/978-94-007 -2129-6_8

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics*, 3(2), 9–24.

Frederick, S. (2005). Cognitive reflection and decision making. *Journal of Economic Perspectives, 19*(4), 25–42. https://doi.org/10.1257/089533005775196732

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational Studies in Mathematics, 67*(3), 237–254. https://doi.org/10.1007/s10649-007-9100-6

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (Vol. 7, pp. 234–283). American Mathematical Society.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.

Heintz, B. (2000). Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin. Springer.

Holm, S. (1979). A simple sequentially rejective multiple test procedure. *Scandinavian Journal of Statistics, 6*(2), 65–70.

Kempen, L. (2018). How do pre-service teachers rate the conviction, verification and explanatory power of different kinds of proofs? In A. J. Stylianides & G. Harel (Eds.), *Advances in mathematics education research on proof and proving: An international perspective* (pp. 225–237). Springer. https://doi.org/10.1007/ 978-3-319-70996-3_16

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405.

Ko, Y.-Y., & Knuth, E. J. (2013). Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors. *The Journal of Mathematical Behavior, 32*(1), 20–35.

Kultusministerkonferenz (Ed.). (2012). Bildungsstandards im Fach Mathematik für die Allgemeine Hochschulreife: Beschluss der Kultusministerkonferenz vom 18.10.2012.

Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: A comparison across preservice secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal, 31*(4), 393–418.

Lew, K., Weber, K., & Mejía-Ramos, J. P. (2020). Do generic proofs improve proof comprehension? *Journal of Educational Research in Mathematics,* 229–248.

Malek, A., & Movshovitz-Hadar, N. (2011). The effect of using Transparent Pseudo-Proofs in linear algebra. *Research in Mathematics Education, 13*(1), 33–58.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41–51. https://doi.org/ 10.2307/749097

Mejía Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*(1), 3–18.

Mejía Ramos, J. P., & Inglis, M. (2009). What are the argumentative activities associated with proof? *Research in Mathematics Education, 11*, 77–78.

National Council of Teachers of Mathematics (Ed.). (2000). Principles and standards for school mathematics. National Council of Teachers of Mathematics.

Rowland, T. (2001). Generic proofs: Setting a good example. *Mathematics Teaching,* 177, 40–43.

Sommerhoff, D, & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by school students, university students, and mathematicians in the context of teaching. *ZDM, 51*(5), 717–730. https://doi.org/10.1007/s11858-019-01039-7

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306–336.

Weber, K. (2013). On the sophistication of naïve empirical reasoning: Factors influencing mathematicians' persuasion ratings of empirical arguments. *Research in Mathematics Education, 15*(2), 100–114.

FORMAL AND LINGUISTIC BREACHES OF CONVENTION IN WRITTEN STUDENT PROOFS

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Learning how to prove is difficult. Especially at the beginning of their studies, students may have difficulties with the mathematical language, but also with the academic language or the formal presentation of a proof. To investigate this, we analyzed 124 students' attempts at proofs from 34 linear algebra submissions using qualitative content analysis. The analysis aimed to identify potential linguistic and formal breaches of convention. Nearly all submissions contained breaches at the level of academic language, mathematical language and at the level of the proof structure. One reason for this may be that the proofs students see in their lectures may not be model proofs. Implications for future research are discussed.

INTRODUCTION

Learning how to prove is often considered one of the main challenges for mathematics students at the beginning of university. Moore (1994) describes understanding and using mathematical language and notation as one of seven major difficulties students have when they do proofs. Other studies confirmed that students struggle with mathematical language and notation (e.g., Lee & Smith, 2009), as well as academic language (Lew & Mejia-Ramos, 2015, Guce 2017). To be able to help students to learn how to do proofs, it is important to identify what a student's proof should look like especially at the beginning of university and what problems students have regarding a written proof presentation. This requires, among other things, to identify the linguistic and formal breaches of conventions that students do in their attempts at proofs.

THEORETICAL BACKGROUND

Written attempts at proof are only considered acceptable proofs if they are approved as such by the mathematical community (Manin, 2012). This assessment is based on implicit socio-mathematical norms and rules (Thurston, 1995), which can differ greatly depending on the situational context (Lew & Mejia-Ramos, 2020). Writing guides such as Beutelspacher (2009), Kümmerer (2016) and Vivaldi (2014) explain norms and rules for writing proofs in German and English. Some of these focus specifically on student proofs, which we define as proof written by a first-year student in an exercise or written exam.

In addition to these writing guides, researchers identified criteria for grading student proofs. For example, Moore (2016) conducted a small expert survey ($N = 8$) to identify these. He found criteria which refer to use both mathematical and academic language *fluently,* and to present the proof with *clarity*. Clarity refers to the explicit statement of arguments or reasons, the use of mathematical language in a familiar way,

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 168–175). PME.

and the organization of the proof to make it readable. Especially being able to organize a proof seems to be important because when students understand how to organize a proof, they can make better use of their cognitive resources to find a proof idea (Selden & Selden, 2013). Ottinger et al. (2016) present five criteria for the formal quality of proofs based on existing research on students´ difficulties with mathematical notation and symbols, for example *using logical symbols* correctly. They analyzed the relationship between the formal quality of student proofs and their quality of argumentation and found that students often struggle for example with the *use of quantifiers*, and with *explicating definitions*. Furthermore, the quality of arguments was correlated with the correct use of logical symbols in proofs (Ottinger et al., 2016). This seems to be important, because Guce (2017) found that of the nine categories she analyzed, *misuse of mathematical symbols* was one of the most common writing convention breaches in students' attempts at proof, surpassed only by *incorrect grammar*. Lew and Mejía-Ramos (2015) also identified potential breaches of convention based on student proofs of an introductory proof course and found fourteen categories of potential breaches, like *lacks proper grammar and punctuation*, *uses non statements*, *uses unclear referents*, and *uses lay speak*. In contrast to Guce (2017) and Ottinger et al. (2016) they also identified categories related to the organization of a proof, such as *fails to make the proof structure explicit*. But they did not examine the frequency of their breaches. Instead, Lew and Mejía-Ramos (2019) investigated how students and lecturers rate those breaches of convention in student proofs in a further study. They identified that lecturers and students rated the significance of these breaches differently. For example, convention breaches in academic language were considered more serious by lecturers than by students (Lew & Mejía-Ramos, 2019). Despite this, breaches of convention in academic and mathematical language and notation are not necessarily negatively assessed or commented on when lecturers grade students' attempts at proof (Moore, 2016). Additionally, lecturers apply less strict conventions to their own blackboard proofs in terms of academic language and proof structure than to student proofs (Lew & Mejía-Ramos, 2020). As such, some students may not be aware of the conventions that their proofs should follow and therefore may not be able to meet them (Thurston, 1995).

Overall, some possible breaches of convention for student proofs have already been identified in the literature to date. However, the various authors emphasize that the categories identified are not yet sufficiently defined. Therefore, we want to look at linguistic and formal breaches of conventions in student proofs, especially regarding academic and mathematical language as well as the structure of a proof.

RESEARCH QUESTION

The focus of this article is to examine the linguistic and formal breaches of convention in students' attempts at proof. The research question addressed is:

What are linguistic and formal breaches of convention that occur in German student proofs (first-year students' attempts at proof) in linear algebra and how frequently do they occur?

METHODS

It is important to mention that we analyze breaches of conventions, that are not necessarily errors. Therefore, a student's attempt at proof with breaches of conventions can still be accepted as a proof and receive full marks in an exercise or exam.

A qualitative content analysis (Mayring, 2015) was conducted to examine breaches of formal and linguistic convention in 124 student proofs from 34 submissions. The attempts at proofs were voluntarily submitted by groups of up to three students in the fourth week of a linear algebra course for teaching profession (second semester, primary or lower secondary school). Students could receive a bonus for their final exam if they submitted their exercise solutions. The analyzed proofs are from the subject area of real matrices and consist of four independent subtasks, for example:

> Prove the following statement. Let $A \in \mathbb{K}^{m \times n}$ and $t \in \mathbb{K}$, then $(tA)^T = tA^T$.

One way to prove this is:

> Proof: Let $A \in \mathbb{K}^{m \times n}, A := \left(a_{ij}\right)$ and $t \in \mathbb{K}$.
> It is $(tA)^T = \left(ta_{ij}\right)^T = \left(ta_{ji}\right) = t\left(a_{ji}\right) = t\left(a_{ij}\right)^T = tA^T$. Hence the claim follows.

The students knew all necessary information for this proof from the lecture notes, for example that $\mathbb{K}$ symbolizes an arbitrary field.

For the qualitative content analysis, we created a category system deductively and chose *clarity* and *fluency*, which were identified by Moore (2016), as super categories for the breaches of convention in student proofs. In those super categories we added categories and associated subcategories based on previous studies (Lew & Mejia-Ramos, 2015, 2019, 2020; Moore, 2016; Guce, 2017, Ottinger, Kollar & Ufer, 2016) and German and English writing guides (Beutelspacher, 2009; Kümmerer 2016, Vivaldi, 2014). While coding the student proofs the category system was revised inductively. For example, we noticed that many students did hand in proofs with only symbolic expressions and no additional words, which is described as a "bad example" of a proof by Vivaldi (2014). Therefore, we added the category *purely symbolic representation* (C21). Afterwards, over 50% of the student proofs were analyzed by a second rater. The coding manual was again refined, and a consensus procedure was carried out. The final category system comprises 21 categories in clarity and 12 categories in fluency as well as additional subcategories. To provide an insight, we describe parts of the category system in more detail.

An example for a category of fluency is *incorrect use of symbols or mathematical notation* (F4, Guce, 2017). This category is coded, for example, when an equal sign is missing in equations or students wrote $A = a_{ij}$ instead of $A = (a_{ij})$. We also coded this category, when mathematical symbols were used in such a way that the common reading of the symbols resulted in a grammatically incorrect sentence (Lew & Mejía-Ramos, 2020). However, all other breaches in grammar, punctuation, and spelling regarding the academic language (Lew & Mejía-Ramos, 2015, 2019, 2020) were coded as *breaches of convention in academic language* (F2, for example incorrect punctuation or or grammatically incorrect sentences).

A category of clarity is breach of convention in *presenting the structure of the proof* (C8, Lew & Mejía-Ramos 2015, 2019, 2020). After coding the first round, we realized that we needed to subdivide this into subcategories in order to get a more accurate picture. We *distinguish between aspects of proof structure that students did not present (M1-M5) and inappropriate representation of proof structure (I1-I4). Part of the proof structure is to mark the beginning (M1) and ending (M4) of the proof with a symbol or a sentence* (Beutelspacher, 2009; Kümmerer, 2016; Vivaldi, 2014). It can also be seen as a breach of convention when the assumptions (M2; Lew & Mejía-Ramos 2015, 2019, 2020) are not stated in the proof (or at least near the proof). In addition, a part of the proof structure is missing, when arguments are not connected logically (M5) either verbally (Lew & Mejía-Ramos, 2015, 2019, 2020) or with a symbol. For example if somebody wrote "t is an element of $\mathbb{K}$. $\left(ta_{ij}\right) = t(a_{ij})$." we coded M5 because a logical connection like "As such" or "$\Rightarrow$ " is missing. Inductively, we also included a subcategory when we could not separate the proof steps of a student's proof at all (M3).

Besides not presenting parts of the proof structure it can also be presented inappropriately. At first, a written proof should reflect *linearity* (I1,Kümmerer, 2016). Therefore, a breach of convention is when the written arguments don't build on each other or don't lead to the proposition being proved (I13; Konior, 1993; Beutelsbacher, 2009, Kümmerer, 2016). Additionally, proofs should not contain inserts (I11), like well-known definitions (Lew & Mejía-Ramos, 2015, 2019, 2020) or duplications (Kümmerer, 2016). Inappropriate references (I12) that do not correspond to a linear structure, like linking of arguments by (normal) arrows (Kümmerer, 2016), are also coded as a breach of convention. Students also sometimes stated inappropriate assumptions (I22, added inductively), for example $A \in \mathbb{R}^{m\times n}$ instead of $A \in \mathbb{K}^{m\times n}$, or presented arguments in a way that is inappropriate to its original logical status (I4, added inductively), for example saying "it is $(tA)^T = tA^T$" instead of "we have to prove $(tA)^T = tA^T$".

RESULTS

We give an overview of the breaches of convention which occurred in the students' attempts at proof and then look at one proof of a student in detail.

Nine categories of breaches of convention were found to occur in at least 50% of the 34 submissions (Tab. 1). Nearly all students handed in at least one proof with breaches of convention in the academic language (F2), used *symbols or mathematical notation* (F4) incorrectly or had breaches of convention in the *presentation of the proof structure* (C8).

	Clarity					Fluency			
Categories	C2	C8	C9	C18	C21	F2	F3	F4	F5
Frequencies	19	34	18	20	23	34	23	30	19

Table 1: Identified breaches of linguistic and formal conventions in over 50% of the students' submissions, results in absolute frequencies, $Max = 34$.

Notes: C2- Mixture of mathematic notation and continuous text, *C8*- Breach of conventions in the presentation of proof structure, *C9*- Abbreviations and mathematical slang, *C18*- Inappropriate use of the subjunctive, *C21*- Purely symbolic representation, *F2*- Breach of conventions in academic language, *F3*- Term not introduced, *F4*- Incorrect use of symbols or mathematical notation, *F5*- Incomplete phrases.

Most of the students who had a breach in the *academic language* (F2) used the punctuation incorrectly, but many also wrote grammatically incorrect sentences. Furthermore, some students did not write *complete sentences* (F5):

"It applies to $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{R}$." (submission 12, translated)

Also, very often the submissions contained breaches of convention in the use of *mathematical notation or symbols* (F4). For example, students repeatedly wrote $A = a_{ij}$ for a matrix instead of $A = (a_{ij})$. All submission also had at least one breach of convention in the *presentation of proof structure (C8). Therefore, we explain this category in more detail and distinguish between missing aspects of proof structure (M1-M5) and inappropriate representation of proof structure (I1-I4). In Tab. 2 an overview of* breaches of conventions regarding the proof structure is given.

	Missing aspects of proof structure					Inappropriate representation of proof structure						
Categories	M1	M2	M3	M4	M5	I11	I12	I13	I21	I22	I3	I4
Frequencies	32	26	8	17	24	21	6	18	14	15	8	14

Table 2: Identified breaches of convention regarding proof structure conventions, results in absolute frequencies, $Max = 34$.

Notes: M1- start of the proof not marked, M2- assumption missing, M3- proof steps not separated, M4- end of proof not marked, M5- lack of logical connectives, I11-Includes inserts, I12- Inappropriate reference, I13- Proof steps do not build on each other, I21- assumptions not marked, I22- inappropriate assumptions, I3- assertions not marked, I4- inappropriate logical status of arguments

The students often did not make the proof structure explicit, for example they often did not *mark the beginning* (M1) or the *end* (M4) of their proof with a symbol or a sentence. The assumptions (M2) were also often not made explicit, or the students did not connect their arguments with logical connectives (M5), like verbal connectives "therefore", "which implies", or symbolic connectives like "⇒ ".

In addition to missing aspects of the proof structure they also did present the structure inappropriate. Almost all submissions included proofs that did not have a linear structure (I1). Many students included comments or insertions (I11) which were inappropriate for this proof, for example:

"The Inverse of the matrix A^T can also be written as A^{-T}." (submission 30, translated).

This comment is not necessary for the proof and the student did not use A^{-T} afterwards. Some students also included inappropriate references (I12) that interrupted the linear structure of their proof, for example arrows that refer to earlier arguments. In addition, the proof steps of students' proofs often did not build on each other (I13).

An example of a student's proof for the task described above is (submission 14; translated):

$$A = a_{ij} \qquad A^T = a_{ji} \qquad t \text{ is constant}$$

$$\left(t \cdot a_{ij}\right)^T = t \cdot a_{ji}$$

$$tA^T = t \cdot a_{ji}$$

In several places this student used *mathematical notation incorrectly* (F4), for example it is $A = (a_{ij})$ instead of $A = a_{ij}$. In addition, the student mostly uses *symbolic expressions* (C21). Regarding the proof structure, the student did not mark the *start* (M1) or the *end* (M4) of the proof. The student did not give an *assumption* like "let $A \in \mathbb{K}^{n \times m}$, $A \coloneqq (a_{ij})$ and $t \in \mathbb{K}$" either (M2). In addition, there are no logical connectives (M5). Finally the proof steps do not built on each other (I13), because the second and the third line are not connected. Nevertheless, the student's idea of the proof is comprehensible.

Some of the categories we identified out of literature were not coded in this study. For example, no student used *unclear references* (C3, Lew & Mejía-Ramos, 2015, 2019, 2020) or *inappropriate naming of variables and constants* (C12, Kümmerer, 2016).

DISCUSSION

We identified linguistic and formal breaches of convention in German students' attempts at proof. A total of 33 categories divided in clarity and fluency (Moore, 2016) and with additional subcategories were identified out of literature and while analyzing students' proofs.

As all students' submissions contain breaches of convention in *academic language* (F2), and more than half of the submissions have breaches of convention in the category of *incomplete sentences* (F5), we can confirm Guce's (2017) results. These frequencies may occur, because students think that breaches in academic language are of little importance (Lew & Mejia-Ramos, 2019). In addition, these breaches are often neither commented on nor assessed by lecturers when grading proofs (Moore, 2016). Additionally, not much attention is necessarily paid to correct spelling and punctuation on blackboard proofs from the lecturers (Lew & Mejia-Ramos, 2020). This could have formed or strengthened the students' belief, that breaches of academic language in proofs are of little importance.

In the category of *incorrect use of symbols or mathematical notation* (F4), almost all submissions showed breaches of convention. A reason for this could be that students seldom receive detailed feedback on their use of symbolic language and mathematical notation (Moore, 2016). Again Guce (2017) had similar results. This is alarming, as

Ottinger, Kollar and Ufer (2016) found that the use of logical symbols is related to the quality of arguments, and knowing mathematical terminology is also relevant for understanding mathematics in general (e.g., Thurston, 1995).

Regarding formal-structural aspects of a proof, breaches of convention in the presentation of the structure of the proof (C8) occurred in all submissions. After taking a closer look, we realized that many students miss to present their proof structure explicitly or do so inappropriately. This may also be due to the fact that proofs in lectures cannot necessarily be regarded as model proofs in terms of structure (Lew & Mejia-Ramos, 2020), and therefore students are not yet aware of the necessary socio-mathematical norms of proof structure (Thurston, 1995). Knowing how to present the proof structure adequately is not only important for students to write an acceptable proof, but also because it can free up cognitive resources while finding a proof idea (Selden & Selden, 2013).

All in all, we see that students' proofs show breaches in academic language, mathematical language, and proof structure. However, some of the identified categories were not coded in the students' proofs we analyzed. Nevertheless, those categories remain in the category system for now because those breaches of convention may occur only in other proofs. For example, inappropriate naming of variables and constants (C12, Kümmerer, 2016) is more likely to happen if the needed variables are not given in the task. In addition, only a very small sample has been evaluated so far, and there may also be other breaches of convention in different areas of mathematics which require further studies. It cannot be ruled out that the students in this study copied solutions from other students and thus reproduced errors. In addition, it is possible that we have also obtained high frequencies in the categories use of *mathematical notation and symbols* (F4) and *academic language* (F2) because the categories are not yet sufficiently finely structured. It will be our task in further research to distinguish them further. In addition, we want to interview experts in order to determine which of the identified breaches of convention are considered to be particularly important by them, especially in student proofs, before the necessary support measures can be developed.

References

Beutelspacher, A. (2009). *Das ist o.B.d.A. trivial!* [this is without loss of generality, trivial!] (9th ed.). Vieweg & Teubner. https://doi.org/10.1007/978-3-8348-9599-8

Guce, I. K. (2017). Mathematical Writing Errors in Expository Writings of College Mathematics Students. *International Journal of Evaluation and Research in Education*, *6*(3), 233-242.

Konior, Jan (1993): Research into the construction of mathematical texts. In: *Educ Stud Math* 24 (3), 251–256. https://doi.org/10.1007/BF01275425.

Kümmerer, B. (2016). *Wie man mathematisch schreibt: Sprache – Stil – Formeln [*How to write mathematically: language-stile-formulae]. Springer Spektrum. https://doi.org/10.1007/978-3-658-01576-3

Lee, K., & Smith III, J. P. (2009). Cognitive and linguistic challenges in understanding proving. In F. Lin, F. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proc. of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 21-26).

Lew, K. & Mejía-Ramos, J. P. (2015). Unconventional uses of mathematical language in undergraduate proof writing. In T. Fukawa-Connelly, N. Engelke Infante, K. Keene & M. Zandieh (Eds.), *Proc. of the 18th Annual Conference on Research in Undergraduate Mathematics Education*, (pp. 201-215). Pittsburgh, PA.

Lew, K., & Mejía-Ramos, J. P. (2019). Linguistic conventions of mathematical proof writing at the undergraduate level. *Journal for research in mathematics education*, *50*(2), 121–155. https://doi.org/10.5951/jresematheduc.50.2.0121

Lew, K., & Mejía Ramos, J. P. (2020). Linguistic conventions of mathematical proof writing across pedagogical contexts. *Educational Studies in Mathematics*, *103*(1), 43-62. http://dx.doi.org/10.1007/s10649-019-09915-5

Manin, Yu. I. (2010): *A Course in Mathematical Logic for Mathematicians*. Springer-Verlag New York. https://doi.org/10.1007/978-1-4419-0615-1

Mayring, P. (2015). Qualitative Content Analysis: Theoretical Background and Procedures. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education* (pp. 365-380). Springer.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266. https://doi.org/10.1007/BF01273731

Moore, R. C. (2016). Mathematics professors' evaluation of students' proofs: A complex teaching practice. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 246-278. http://dx.doi.org/10.1007/s40753-016-0029-y

Ottinger, S., Kollar, I. & Ufer, S. (2021). Content and form - all the same or different qualities of mathematical arguments? In C. Csíkos, A. Rausch & J. Szitányi (Eds.), *Proc. of the 40th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 19-26). PME.

Thurston, W. P. (1995). On proof and progress in mathematics. *For the learning of mathematics, 15*(1), 29-37.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematics Enthusiast*, *10*(1), 303-334. https://doi.org/10.54870/1551-3440.1269

Vivaldi, Franco (2014): *Mathematical Writing*. Springer London. https://doi.org/10.1007/978-1-4471-6527-9

NAVIGATING THE JOURNEY FROM PROFESSIONAL DEVELOPMENT TO THE CLASSROOM: ACCOUNTABLE TALK IMPLEMENTATION IN PRIMARY MATHEMATICS CLASSES

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Orchestrating high-quality dialogic discourse in primary mathematics classes is a considerable challenge for teachers. Research on the design and impact of professional development (PD) programs aimed at this challenge is limited. In particular, studies examining the trajectory of change in teachers' quality of discourse during and after PD programs are scarce. Our research focused on a specific type of discourse named Accountable Talk (AT). In this report we describe the cases of two mathematics teachers who participated in a PD program around AT. We followed them into their classes, to assess the impact of the PD on the quality of the discourse. Findings point to factors that may affect teacher learning and implementation of AT.

INTRODUCTION

Research on the quality of discourse in mathematics classrooms has accumulated in the past two decades, in light of the general recognition that the quality of discourse - measured according to social and epistemic criteria - impacts on various aspects of student learning (Resnick et al., 2015). However, maintaining high-quality discourse poses a considerable challenge for teachers (Schwarz & Baker, 2016). Supporting teachers in acquiring skills to facilitate high-quality discussions in their mathematics classes is a multifaceted endeavor. The literature on the design and impact of such support is still underdeveloped. To address this gap, a five-year research project was implemented, with the goals of (a) designing a PD program aimed at promoting high-quality discourse in primary mathematics classes; and (b) exploring processes of professional learning and development of teachers participating in the program. At the heart of the PD was a specific type of discourse, named *Accountable Talk* (AT) (Michaels et al., 2008; details follow). To train teachers to use this discourse in their classrooms, a video-based design was utilized, with clips demonstrating AT teaching moves. Alongside participation in the PD, the teachers were encouraged to implement AT in their teaching. The study examined both settings - the PD and the teachers' classrooms - with particular focus on trajectories of change that the teachers have undergone throughout the PD year. Additionally, to assess the degree to which teachers preserved AT moves and practices, the study included a follow-up in the teachers' classrooms about one year after the PD's conclusion. We report herein on the study of two cases, focusing on two research questions: (RQ1) How can trajectories of change in the teachers' talk practices be characterized? (RQ2) To what degree was the implementation of the learnt practices sustained, if at all, after one year? In the

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 176–183). PME.

following, we provide a short theoretical background concerning the two core elements combined in our study: AT and video-based PDs.

THEORETICAL BACKGROUND

Michaels et al. (2008) defined AT as an argumentative dialogic discourse that takes place within a collaborative learning environment through which students learn to reason in the context of a field of knowledge. This kind of discourse is characterized by three dimensions of accountability: to the community, to reasoning and to knowledge. Accountability to the community means that the teacher acts to involve all participants in the dialogue as agents. Accountability to reasoning signifies that participants utilize logical moves or present evidence for constructing knowledge. Accountability to knowledge indicates that, in their arguments, students rely on knowledge already constructed and agreed upon. AT offers a set of talk moves such as “press for reasoning”, “say more”, and “solicit additional viewpoints”, which provide teachers with tangible means for eliciting students’ reasoning, as well as for holding students accountable to making their thinking understandable to others (Resnick et al., 2015). AT intermingles epistemic and social aspects, thus teachers who are not familiar with AT need to first experience it in a supportive learning environment around their field of knowledge. This is crucial especially since AT, and dialogic teaching in general, differs from traditional methods that many teachers experienced as students (Ball & Forzani, 2010). However, despite the accumulating evidence affirming the effectiveness of dialogic teaching, and the substantial resources dedicated to PD programs around it, its implementation in schools remains limited (Schwarz & Baker, 2016). While research indicates the challenging nature of shifting teachers' practices to embrace innovative teaching methodologies (Heyd-Metzuyanim et al., 2019), recent efforts have been undertaken to delve into the intricacies of the transition from theory to practical implementation (Baor, 2021). Among the methods that have proven successful in supporting teacher learning towards implementing innovations are video-based PD programs (Coles et al., 2019). A video-based PD enables a microanalysis of teaching practices as a part of collective reflection (Karsenty & Arcavi, 2017), and consecutive video-based sessions alternating with teaching sessions support the development of plans to improve practices (Borko et al., 2011). In addition, general guidelines recommended in the literature for effective PD include situating the PD in teachers' actual classroom realities while inviting collaborative reflection on authentic episodes, and also designing the PD as a long-term undertaking (e.g., Putnam & Borko, 2000). As we show next, our PD design (constituting the first phase of the study) relied on these guidelines, enabling a solid ground to explore teachers’ learning and implementation trajectories.

METHOD

Setting, participants and data sources

The data for the larger project, including the two cases reported here, were collected in 2016-2018. A 30-hour elective PD program, centering on AT, was offered to primary mathematics teachers in two cohorts located in two different cities. The PD was spread over 10 sessions along one schoolyear and recognized for professional accreditation by the Ministry of Education. In total, 45 teachers participated in the PD, with a range of 3-23 years of experience. All PD sessions were videotaped and transcribed. Eight teachers were selected for an in-depth investigation, based on purposive sampling to represent diverse backgrounds. For these teachers, classroom data were gathered in 3 points during the PD year: before, halfway through and at the end of the PD. At each time point a whole lesson was videotaped in each teacher's class. A fourth lesson was videotaped about a year after the end of the PD. In addition, video-based Stimulated-Recall Interviews (SRIs) were conducted with each teacher after every recorded lesson. In this paper, we focus on two of the eight teachers, Jill and Eda (pseudonyms). Jill was a novice teacher with 3 years of teaching experience, a graduate of the mathematics track in a teacher education college. Eda was a veteran teacher with 23 years of teaching experience. As a graduate of a humanities track, she had no formal training in mathematics. Both teachers were not enrolled in mathematics education PDs before.

Data analysis

Analysis of patterns of teachers' participation in the PD. Six PD sessions were analyzed to determine patterns of participation for each of the 8 selected teachers. We coded the *level* of participation (the number and length of the teacher's utterances) and the *quality* of participation (explorative, semi-explorative or superficial, reflecting the teacher's reliance on reasoning or evidence when making observations).

Analysis of the implementation of AT in the teachers' classes. Two existing coding schemes were used: IQA (Instructional Quality Assessment; Boston, 2012), that rates the level of cognitive demand and the level of AT on a scale of 1 to 4, and ATC (Accountable Talk coding; Heyd-Metzuyanim et al., 2019), a tool for coding AT moves during a lesson. Due to space limitations, we present here only the IQA analysis for the two teachers. The full details on IQA categories, and how a rate of 1-4 is obtained from classroom observations, can be found in Boston (2012). In addition, specific rubrics were designed for coding the types of questions asked by the teacher (open/closed), the type of student answers (justified/other), and the amount of evaluative feedback.

Analysis of SRIs. Coding of SRIs employed a specially-developed rubric, comprising (1) the quality of the teacher's participation (explorative/semi-explorative/superficial, see above); (2) mathematical aspects (i.e., to what degree did the teacher refer to mathematical issues such as solution methods, mistakes, etc.); (3) dialogical aspects (to what degree did the teacher refer to social-dialogic issues such as collaboration or respectful discourse); (4) indications of the teacher's orientations about what is desired teaching (i.e., facilitation of knowledge construction vs. knowledge transmission).

Based on the various means of coding, a quantitative and qualitative summative analysis was performed using methods of descriptive statistics and Micro-scale analysis, to identify what changes, if any, occurred throughout the PD year. Individual attributes such as mathematical background and teaching experience were taken into account in the analysis, to assess their prominence as possible indicators of change.

FINDINGS

Teachers' pre-PD orientations

Initial differences between the two teachers' orientations were found in the individual interviews prior to the PD beginning. As the quotes below show, Jill and Eda seemed to have different views regarding desirable discourse practices in the mathematics class: (notation: 'Ia-x' signifies Interview #a, transcript line #x).

> You should embark on an inquiry question […] Give them a task to handle, then, let's talk about it together, let's present ways to solve it. (Jill, I1-44)

> First of all, it shouldn't be […] a Ping-Pong dialogue between me and a student. First I need to ask a question, to trigger their thinking […] and have them start talking. […] It should reach a state where they talk to each other - 'you said that, but I don't think so, I think otherwise', something like that. (Jill, I1-60)

> All these years I taught by imparting knowledge, frontally [...] [Today] I worked for almost two lessons in order to explain them, [...] I tried to show them, I brought pencils, I brought coins [...] they had a really hard time understanding what I want. (Eda, I1-16,48)

> [A good lesson is when] disruption is minor, I'm not taken out of focus […] and they succeed in arriving at the conclusions that I am trying to lead them to. (Eda, I1-60)

Jill's orientations appear to align with an approach of 'facilitating knowledge construction', emphasizing collaborative inquiry and recognizing the significance of student interaction in developing thinking. Eda's orientations seem to align more with a teacher-centered approach of 'knowledge transmission', although she also ascribes importance to having students arrive at conclusions, in a teacher-led process.

Teachers' patterns of participation in the PD

The teachers differed in the level and quality of their PD participation. In the 6 coded sessions, Jill spoke more than Eda (44 and 7 utterances, respectively), with longer utterances (average of 19.8 and 16.4 words per utterance, respectively). Jill's share of speech coded as explorative was higher than Eda's (56% and 14%, respectively). Below is an example of an explorative utterance by Eda, about a videotaped episode:

> I liked it that a child explains to a child, that the girl explained to him, and he asked her. He didn't ask the teacher [...]. I think the gain of a child explaining to a child is more than when the teacher explains, they know how to explain each other sometimes better than us.

Teachers' patterns of participation in the SRIs following lessons 2,3

The analysis of the two teachers' patterns of participation in the two SRIs following lessons 2 and 3 (aggregated together) is summed in Table 1. As can be seen, in all 4 categories (detailed earlier), substantial differences were found between Jill and Eda.

Type of participation	Jill	Eda
(1) Quality of participation: % of explorative utterances	96%	31%
(2) Mathematical aspects: % of the utterances referring to mathematical issues	67%	16.2%
(3) Dialogical aspects: % of the utterances referring to social-dialogic issues	33%	83.8%
(4) Teaching orientation:		
% of utterances indicating ‘knowledge construction’	98%	13.2%
% of utterances indicating ‘knowledge transmission’	2%	86.8%

Table 1: Analysis of the two teachers’ SRIs following lessons 2 and 3**.**

The examples below illustrate differences found between the teachers in Category (4).

Example 1. Jill reacts to an observed episode from her class. The discussed issue was the equivalence between thirty coins of 10 cents and three dollars:

> It's basically the same thing, but they don't understand each other, so at this point I open the question to the whole class, both for making them partners and because it hands it back to them, so it helps them understand that they did the same thing and thought the same thing. […] Each one looked at it differently, but it’s okay, it’s the same thing. (Jill, I2-46)

Example 2. Eda reacts to an observed moment from her videotaped lesson, showing her summing up what was learned so far:

> Constantly to summarize for them, to arrange for them, to illustrate for them, that’s something that I very much work on now. (Eda, I3-17)

Jill’s quote displays a view of herself as a discussion facilitator, whose role is to foster collaboration and transfer authority over knowledge construction to students, as seen in her words ‘I open the question to the whole class’, ‘making them partners’, ‘hands it back to them’. Eda’s quote illustrates a teacher-centered perspective, underscoring a view of the teacher as a knowledge provider. This differs from the PD's agenda about the importance of collaborative student-led summaries guided by the teacher.

Teachers’ change over time in implementing AT

The IQA analysis for the two teachers is summarized in Figure 1, showing the average ranking of IQA components in the four lessons of each of the teachers.

As clearly seen, Jill showed notable, although non-linear, improvement in IQA scores through the 3 lessons filmed during the PD year, which continued in lesson 4 after about a year from the PD completion. Eda’s IQA scores improved between lessons 1 and 2 but declined in lesson 4 after a year, to a level similar to her starting point.

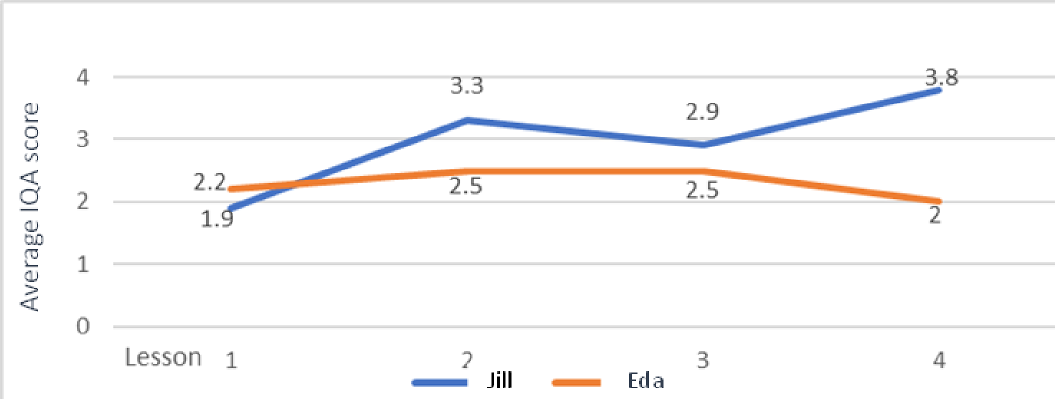

Figure 1: Average score of IQA components in four lessons of the two teachers.

Figure 2 presents the analysis of the teachers' question types and evaluative feedback as coded for each of the four lessons.

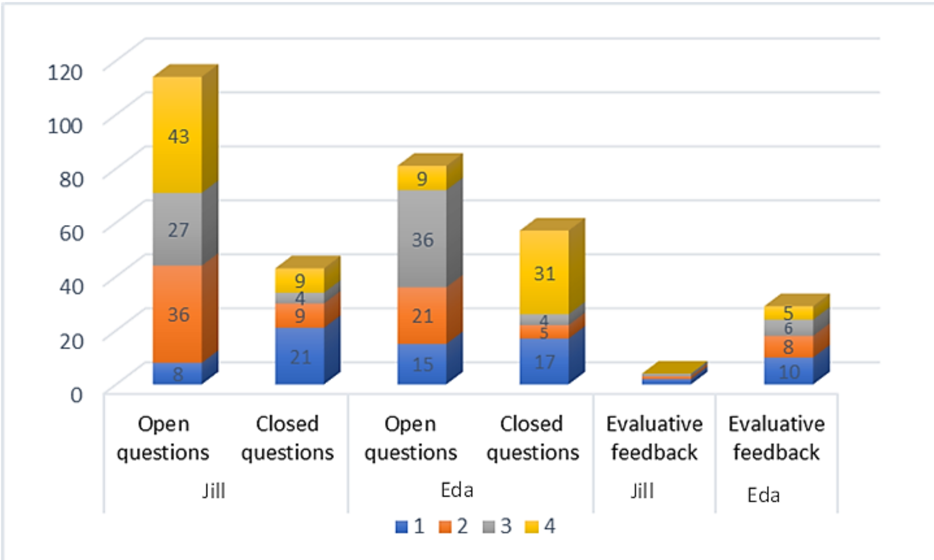

Figure 2: Question type and evaluative feedback in four lessons of the two teachers.

While in lesson 1 both teachers favored closed questions over open ones (in Jill's case this difference was more salient), in lessons 2 and 3 they shifted towards more open questions. However, whereas Jill maintained this tendency in lesson 4, Eda reverted to a higher number of closed questions. Figure 2 also displays the amount of evaluative feedback. Both teachers consistently reduced their use of evaluative feedback across the four lessons, with Jill maintaining minimal use from the start.

Figure 3 presents the types of student answers in lessons 1-4 of both teachers. In lesson 1, students in both classes provided fewer justified answers compared to other answers. Lesson 2 shows an increase in justified answers, yet while in Jill's class justified answers outnumber other kinds of answers – a tendency sustained in lessons 3 and 4 - in Eda's class a smaller number of justified answers remained across all lessons.

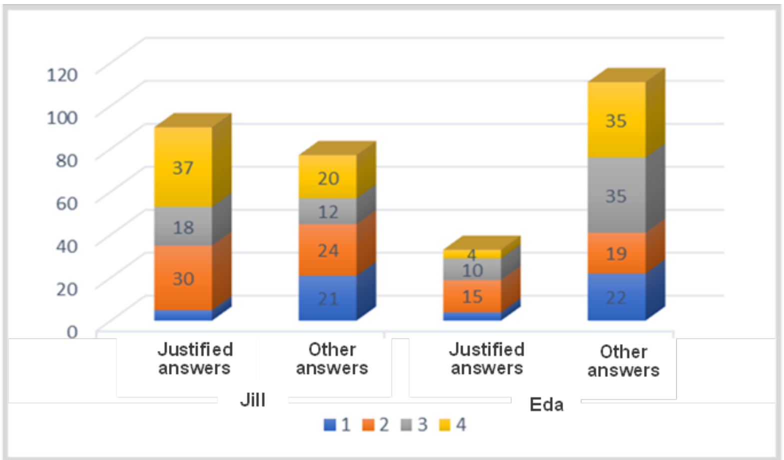

Figure 3: Number of justified and other answers in four lessons of the two teachers.

DISCUSSION

The findings unveil distinctly different learning trajectories: Jill's journey reflects a notable and sustained improvement in implementing AT over time, while Eda's path shows limited change. We examine possible factors that may explain this disparity.

Personal background. Jill majored in mathematics education, while Eda lacked formal mathematical training. Since mathematical discourse according to AT combines epistemic (mathematical) and social aspects, formal training in mathematics teaching may be a crucial factor in effective implementation of AT practices in mathematics classes, which can explain some of the differences we found in the teachers' trajectories. Eda's seniority was not necessarily an asset in implementing AT; practices that are deeply ingrained through years of experience can be difficult to change. In contrast, Jill, a young teacher with minimal experience, could more easily use the PD as a catalyst for improving discourse quality. More research is needed to confirm these suggested links between teachers' background and sustainable AT implementation.

Patterns of participation in the PD. The two teachers' engagement in the PD aligned with their degree of implementation improvement, suggesting that the level and quality of PD participation may, to some extent, predict how teachers apply learned practices.

The pedagogical flexibility of the teacher. By 'pedagogical flexibility' we refer to a teacher's openness and capacity to cohere between her perspectives and her actual practice. The SRIs findings revealed differences in the two teachers' degree of explorative participation and references to mathematical and dialogical elements in their lessons. Interestingly, compared to the pre-PD findings, it seems that both teachers remained consistent in their perspectives about teaching; simply put, Eda stayed with the 'knowledge transmission' view and Jill stayed with the 'knowledge construction' view. What has changed for Jill is the gap between her stated teaching orientation and her observed practices: Jill's first lesson was characterized by practices aligned more with knowledge transmission (see Figures 1,2), than with how she described a desired discourse. This misalignment parallels findings in Baor's (2021) study, where a teacher's adoption of an explorative pedagogical approach did not fully translate into her actual classroom discourse. As Heyd-Metzuyanim et al. (2019) note, teachers' adaptation of actual practices to concur with envisioned ones, is a complex, possibly non-linear process. In Jill's case, the gap gradually narrowed over the year. We speculate that she was flexible enough to allow herself to try out AT moves that she saw in the demonstrative videoclips discussed in the PD. In contrast, Eda's long-held views about teaching, which in a sense conflicted with the PD's agenda, may have reduced her openness to implement the discussed strategies in her class.

Although the two cases we presented cannot describe general trends, they stress the complexity underlining changes (or lack thereof) towards high-level talk practices. A combination of factors such as prior mathematical backgrounds, patterns of PD participation and pedagogical flexibility play a crucial role in the degree of change. It is likely that these factors interrelate and that there are more nuances to be found. Our findings can be seen as another step on the road to understanding how and why a PD program's effectiveness and long-term sustainability varies across individual teachers.

References

Ball, D. L., & Forzani, F. M. (2010). Teaching skillful teaching. *Educational Leadership*, *68*(4), 40-45.

Baor, R. (2021). *Processes of implementation of inquiry teaching.* Unpublished Doctoral Dissertation. The Technion Institute of Technology, Haifa (in Hebrew).

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representation of teaching in practice-based professional development programs. *ZDM, 43*, 175-187.

Boston, M. D. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal*, 113(1), 76–104.

Coles, A., Karsenty, R., Beswick, K., Oates, G., & Abdulhamid, L. (2019). The use of video for the learning of teachers of mathematics. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd PME conference* (Vol 1, pp. 107-135). PME.

Heyd-Metzuyanim, E., Smith, M., Bill, V. (2019). From ritual to explorative participation in discourse-rich instructional practices: a case study of teacher learning through professional development. *Educational Studies in Mathematics*, *101*, 273–289.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of Mathematics Teacher Education*, *20*, 433-455.

Michaels, S., O'Connor, C., & Resnick, L. B. (2008). Deliberative discourse idealized and realized: Accountable talk in the classroom and in civic life. *Studies in Philosophy and Education*, *27*(4), 283-297.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning?. *Educational researcher*, 29(1), 4-15.

Resnick, L., Asterhan, C., & Clarke, S. (2015). *Socializing intelligence through academic talk and dialogue*. American Educational Research Association.

Schwarz, B. B. & Baker, M. J. (2016). *Dialogue, argumentation and education: History, theory and practice*. Cambridge University Press.

EXPLORING PERCEIVED VALUE DIFFERENCE SITUATIONS IN AUSTRALIAN MATHEMATICS CLASSROOMS

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Recent research has highlighted how students might disengage from classroom activities when their values and their teachers' are different. The present study examined 625 secondary school students' perspectives and experiences to better understand what these value difference situations look like. Analysis of the 29 identified instances of value differences revealed the need to propose a new category - mathematical content value – to add to the existing classification of values. The findings reveal an issue where students often struggle to appreciate the value of learning specific mathematical content, even when teachers emphasise its practical usefulness. The results also highlight that teacher' excessive reliance on textbooks runs counter to student' mathematics educational values, hindering their learning.

INTRODUCATION

Previous research (Seah & Andersson, 2015) revealed how students and teachers bring their personal values (that is, regarding what is important) in mathematics education into the classroom. Since individuals' values in mathematics education stem from the nature of mathematics as well as their sociocultural and educational experiences (Seah, 2019), students and their teachers develop and interpret these values in diverse ways. This phenomenon becomes more complex in multicultural societies such as in Victoria, Australia, where teachers, students and families hail from a diverse ethnic and racial profile. It is thus reasonable to assume that interactions in the (Victorian) mathematics classroom is characterised by the coming together of a range of teacher and student values relating to mathematics, mathematics teaching and learning, and education generally.

Given that decisions and actions (in mathematics education) are driven by values (Seah, 2019), the alignment of students' values with their teacher's influences the quality of classroom interactions, and thus, of the lessons. For instance, a teacher who values *group work* would incorporate collaborative activities in their lessons, but any of their students valuing *independent work* might disengage with or avoid such activities. Indeed, students who hold different values from their teachers are likely to resist or disengage, negatively affecting their interest and performance in mathematics learning (Kalogeropoulos, 2016). Effective teaching thus requires an understanding of value differences, as teacher intentions alone are unlikely to ensure successful lessons. This calls for a thorough exploration of the attributes valued by students and their teachers in mathematics pedagogy.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 184–191). PME.

Previous studies assumed that students would express their values through feedback to their teachers (Kalogeropoulos et al., 2021). However, students may conceal their values for a variety of reasons. Even when students have choices, these choices are usually constrained within the classroom context (Clarkson et al., 2019). Thus, to gain a more comprehensive understanding of value differences in the mathematics classroom, it is important to consider whether students are able to contemplate, compare, and express their values during classroom interactions. In this context, the current study seeks to address this gap, establishing a comprehensive overview of the types of value differences in mathematics classrooms, based on Victorian students' perspectives.

The Research Question guiding our study here is: What sorts of values espoused by students and their teachers most often end up in value difference situations?

THEORETICAL FRAMEWORK

Instead of being viewed as an objective body of knowledge awaiting discovery, mathematics has been acknowledged as being socialised knowledge (Bishop, 1988). Research into values and valuing in mathematics education from the late 1980s (see Bishop, 1988, 1996) acknowledges that the discipline and its pedagogy is culture-dependent. Early research literature in this area reflects a conception of values as an affective variable (Bishop, 1996). Seah (2019), inspired by the tripartite model of the human mind, later redefined values as being conative in nature. In doing so, Seah could explain why people often passionately embrace their values and why values can be made visible through decision processes. Hence, this study adopted Seah's (2019) definition of values in mathematics education, in which valuing is concerned with "an individual's embracing of convictions in mathematics pedagogy...[shaping] the individual's willpower to embody the convictions in the choice of actions" (p. 107). Since a value directs an individual's course of action, values can be regarded as being motivational (E, 2023). However, motivation might not fully explain persistence—the character of will and determination embedded in values empower determination amid obstacles (Seah & Andersson, 2015), not just guiding actions but defending them. As such, value difference situations arising from interactions between teachers and students are characterized not just by motivation, but also involve the will inherent in values.

Bishop's (1996) categories of values in mathematics education, namely, general educational, mathematical, and mathematics educational values, provided a useful framework for categorising the types of value differences perceived. Mathematical values relate to the mathematics discipline itself, and they have been identified to be the three complementary pairs of *rationalism* and *objectism, control* and *progress,* and *mystery* and *openness* (Bishop, 1988). On the other hand, mathematics educational values refer to the objects, experiences, and pedagogies that students and teachers consider important to learning and teaching mathematics (Seah, 2019). Last but not least, general educational values cover the moral, ethical, citizenship, and sometimes, religious values that a given educational system aims to impart to its students. Unlike the first two categories of values, general educational values are part of the educational process but are not directly related to mathematics instruction.

METHODS

Data were collected online using the 'What I Find Important Too' questionnaire, accessible at https://www.surveymonkey.com/r/VASstu_v4. The questionnaire draws on hypothetical situations to encourage student respondents to delineate instances where disparities exist between their own values and those emphasised by their teachers during mathematics lessons (E, 2023). Here we focus on the open-ended question posed: What was the value differences situation like?

Participants were chosen via stratified probability sampling of schools across Victoria, ensuring diversity in terms of gender, ethnicity, and educational background. The 625 participants (50% female) were from urban and regional secondary schools located in Victoria. Students self-identified their ethnicities as Australian (349), European (116), Asian (73), North African and Middle Eastern (17), Americans (9), Indigenous Australian (6), Sub-Saharan African (4), and Other (43).

Students' responses were analysed using thematic analysis (Braun & Clarke, 2006). Initial nodes were generated and coded inductively. For example, "I asked her to teach in a way that we could all understand" was coded as *understanding* to reflect this valuing. Subsequently, nodes were organised deductively into value categories, guided by a checklist that included three pairs of mathematical values (Bishop, 1988) and seven pairs of complementary mathematics education values (Kinone et al., 2020). For example, the value nodes *understanding* and *textbook* were categorised as 'mathematics educational values'. The checklist provided flexibility, allowing for the identification of additional value categories.

RESULTS

29 instances of value differences were reported. While 10 (34.48%) of these were either maths educational or general educational values, a majority could not be located within the three value categories which Bishop (1996) identified. Rather, they appeared to relate to mathematical content. Bishop's (1996) three-category system does not seem to adequately describe the range of value differences perceived by secondary school mathematics learners. As a result, an additional category – mathematical content values – is being proposed here.

Mathematical content value differences refer to those incidents when specific mathematical content that is important to the teacher is not important to the students, and vice versa. Of the 29 value differences reported, 19 of these were related to student and teacher differences in valuing particular mathematical content. Specifically, within this category, 11 students referenced they did not find algebra to be important. For instance, one student noted, "It [algebra] is just numbers and letters mixed up together; it is not reality". Five students did not specify the name of content, but they also expressed disagreement with their teachers' view that some specific mathematics knowledge was important, noting for instance, "we were learning something in maths that we are never going to use in everyday life and my teacher said it was important". The rest of the mathematical content value differences pertained to numerals, coordinates, addition/subtraction, geometry, trigonometry, equations, data and stats.

Our data suggested that these mathematical content value differences often stemmed from students' questions about the usefulness of the specific content. The thought process of students revealed that while they would try to assess the value of these mathematical concepts based on their perceived usefulness, this approach often fails to convince them. For example, one student remarked, "we will never use it [algebra] in the future because why would people make people waste their time when there [sic] shopping and try and figure out what the hidden number or something." Another student questioned the importance of trigonometry, stating, "but teacher thinks it is important. How will we use [this] mathematics in day-to-day scenarios?" Interestingly, teachers' justifications for the mathematical content value, as mentioned in students' responses, also emphasised its usefulness in careers, everyday life, and future. One student emphasised, "we were learning something in maths that we are never going to use in everyday life and my teacher said it was important, but I didn't take it seriously because it was useless. She [my teacher] gave me a situation of how we would use this maths content in everyday life, and it was something that would never happen". In this instance, the concerned teacher attempted to address the disparity in values related to mathematical content by highlighting its practical relevance in everyday life, but she was not successful in engaging their students in her endorsement.

Mathematics educational value differences were next most commonly reported by the secondary school participants, where the source of difference rooted in the differing perceptions of teachers and students regarding approaches to mathematics teaching and learning respectively. There were nine of the 29 value differences documented which pertained to the nature of mathematics educational values. Specifically, there were 5 instances which revealed differences in valuing between teachers' teaching relying on textbook questions and students' embracing of other pedagogies. For example, a student pointed out, "My teacher doesn't listen to us and she thinks by doing textbook questions it'll help us, which isn't true". From what a student wrote, "I think that doing the same type of questions all the time is unimportant. My teachers make us repeat them all the time", it appears that students interpreted their teachers' value as repetition unnecessarily. Indeed, other students wrote their maths education values that "I want constant and different types of problems", "I asked her to teach in a way we could all understand and not just do workbook everyday", "I have difficulties taking in information because as a class we just do book work and other teachers use different methods to teach their classes … I told her [my teacher] to use different examples and/or teachings skills" and "It becomes boring to do that many questions, which means we don't concentrate well in class". Notably, these responses reflected Kalogeropoulos's (2016) assertion that students who hold different values from their teachers are likely to resist or disengage from pedagogical activities, negatively impacting their interest and performance in mathematics.

Only one student referred to value differences of the general educational value nature. It involved the differing valuing of *equity*: “We had a deadline and most of us met it, but some didn’t, so the deadline was extended. I thought that was unfair to the people who had worked hard to meet the deadline”. Perhaps one possible explanation is that generally these values are already accepted by the (educational) community, such that the chance of any such value being involved in value difference situations was less likely. In fact, Seah (2019) suggested that general educational values are already being inculcated in students as part of fulfilling the professional and ethical responsibilities of teaching.

DISCUSSION AND CONCLUSION

Although our data about mathematical content value difference situations suggested that some teachers emphasise the importance of mathematical content due to its usefulness, it cannot be definitively categorised as valuing *application*. *Application* as a mathematics educational value refers to “valuing application of mathematics in various problematic situations in mathematics learning” (Kinone et al., 2020, p. 44), which is more aligned with emphasising pedagogies that can enhance students’ abilities to recall factual knowledge and concepts readily and flexibly to find solutions for mathematical questions. However, teachers highlighted the importance of mathematical content in everyday life and future jobs/careers. This emphasis seems to extend beyond mathematical learning and appears unrelated to pedagogy, making it incompatible with the mathematics educational value as defined by Bishop (1996).

In addition to defying categorisation within existing value categories, there are also arguments for acknowledging the mathematical content value category. Current knowledge about values (Seah, 2005) suggested that cultures construct and develop mathematics in different ways, resulting in educational systems that reflect what cultures consider valuable, that is, what they value. In other words, mathematics is socialised knowledge; knowledge that has been cultivated and developed in response to particular needs within cultures, not objective knowledge (Clarkson et al., 2019). This implied that mathematical knowledge is selected and organised to become content knowledge for teaching and learning, a process inherently embedded with values. Illustrating this is the stated aims for revising the Australian mathematics curriculum (ACARA, 2021), which highlighted the need to “remove outdated and non-essential content, add new content that has been identified as important for students to learn, better sequence student learning and give teachers greater clarity and guidance about what they are expected to teach” (p. 1). Although this document does not explicitly state why certain contents are considered important for teaching and learning, what it does highlight is the value attributed to different mathematical content.

This study has emphasised the need to focus on mathematical content values. It is noteworthy to reflect on the fact that prior studies have often approached the body of mathematical knowledge as a whole, emphasising its importance as long as the existence of specific content that can be applied in everyday life or future careers can be justified. However, clear indications suggest that students may encounter difficulties in discerning the value of each piece of mathematical content. This aligns with Atweh et al.'s (2010) observation that students persist in believing that some content is largely meaningless, particularly when teachers cannot demonstrate how all mathematical concepts can be applied to real-life situations. Recently, Rosenzweig et al. (2020) emphasised that a more effective way to increase student utility is to help students think about the value of course material, which seems to suggest a new trend: a return from considering body of knowledge as a fixed whole to a focus on specific content knowledge. Therefore, the urgent next step is to concentrate on mathematical content values and find a way to respond to situations in which students do not see the value of learning particular mathematical content.

In most value difference situations involving mathematics educational values, teachers tend to value practicing problems in textbooks, but students generally do not value this way of learning mathematics. Notably, our findings demonstrate how problematic it can be with teachers' heavy dependence on the mathematics textbooks, as students appeared to associate textbook use with mechanical or habitual repetition. This result may not be surprising given that secondary-level mathematical textbooks are often filled with relatively low-level, repetitious exercises (Stephens, 2019). Rather than repeatedly solving the same types of problems found in the textbooks, students are more likely to seek diverse examples and instructional methods that are aligned with their mathematics educational values. What this implies is that educators and teachers who rely on textbooks need to consider how to meet their students' mathematics educational values by carefully selecting or redesigning textbook problems (e.g., by changing just one or two numbers to extend learning opportunities). Otherwise, these students with conflict values from their teachers would exhibit resistance or disinterest in pedagogical activities.

Inherently, repetitive and low-complexity problems are not necessarily bad, because students learn procedures through sufficient repetition. However, this pedagogy relying on overwhelming prevalence seems to limit students' opportunities to feel and contemplate the value of mathematical content. An example can be found in early research; Goldin (2004) expressed concern over the inclusion of activities related to solving discrete mathematics problems in the secondary school mathematics curriculum. In his words, if these problems were added to standardized achievement tests with the intention of evaluating nonroutine problem-solving skills, then routine methods for solving them would be explained and utilised in numerous parallel practice problems in school workbooks. This means that this particular knowledge is included in the mathematics curriculum as a set of information to be memorised and strategies to be used routinely, which is exactly the aspect that is not valued by students frequently mentioned in mathematics educational value differences situations. More importantly, in this way, the potential of discrete mathematics in terms of triggering attributes such as experimentation, logical reasoning, and problem-solving would certainly be diminished (Goldin, 2004). This seems to reveal the mathematical content value of learning discrete mathematics, but it is obscured by certain pedagogy. It is also worth reflecting on the fact that these attributes have been described in previous studies as characteristics of the mathematical discipline as a whole, rather than analysing the relationship of the individual essential mathematical content to these attributes. The question raised here is whether specific mathematical contents containing these attributes are the precise reason the discipline of mathematics exhibits these characteristics at the macro level. Therefore, it is necessary to unpack the values embedded in each mathematical content.

To conclude, the present study investigated Australian secondary school students' experiences of perceived value differences in mathematics lessons, and identified 29 perceived value differences in the data collected, which revealed the need to add a new category, namely, mathematical content value, to the existing classification. The mathematical content category was reported most frequently in this research study, further indicating an issue: students do not value particular mathematical content. Conversely, the mathematics educational values perceived by secondary students nearly all relate to their teachers' values related to teaching from textbooks. Students in Australia often face struggles when their teachers rely solely on mathematics textbooks for teaching. This is due to the textbooks being filled with repetitive and straightforward problems, which is in conflict with students' mathematics educational values. Thus, the findings of this study could offer teachers inspiration to enhance their pedagogy, aligning it with students' values in mathematics education.

References

Atweh, B., Clarkson, P., & Seah, W. T. (2010). What values do middle school students attribute to studying mathematics: A pilot study. In W. Chang, D. Fisher, C. Lin & R. Koul (Eds.), *Envisioning the future* (pp. 9–20). Curtin University.

Australian Curriculum, Assessment and Reporting Authority. (2021). *What has changed and why? Proposed revisions to the F–10 Australian curriculum: Mathematics.* https://www.australiancurriculum.edu.au/media/7120/ac_review_2021_mathematics_whats_changed_and_why.pdf

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Kluwer Academic Publishers. https://doi.org/10.1007/BF00751231

Bishop, A. J. (1996). *How should mathematics teaching in modern societies relate to cultural values—Some preliminary questions.* Paper presented at the Seventh Southeast Asian Conference on Mathematics Education.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Clarkson, P., Seah, W. T., & Pang, J. (Eds.). (2019). *Values and valuing in mathematics education: Scanning and Scoping the Territory*. Springer International Publishing. https://doi.org/10.1007/978-3-030-16892-6

E, A. (2023). *Value alignment strategies in Australian mathematics classrooms* [MEd thesis]. The University of Melbourne.

Goldin, G. A. (2004). Problem solving heuristics, affect, and discrete mathematics. *ZDM*, *36*(2), 56–60. https://doi.org/10.1007/BF02655759

Kalogeropoulos, P. (2016). *The role of value alignment in engagement and (dis)engagement in mathematics learning* [Doctoral thesis]. Monash University.

Kalogeropoulos, P., Russo, J. A., & Clarkson, P. (2021). Exploring educator values alignment strategies in an intervention context: The emergence of the beacon strategy. *ECNU Review of Education, 4*(2), 327–348. https://doi.org/10.1177/2096531120923127

Kinone, C., Soeda, Y., & Watanabe, K. (2020). The influences of teacher valuing on the development of student valuing in mathematics education [in Japanese]. *Journal of Japan Academic Society of Mathematical Education, 26*(1), 43–58. https://doi.org/10.32296/jjsme.95.rs_105

Rosenzweig, E. Q., Wigfield, A., & Hulleman, C. S. (2020). More useful or not so bad? Examining the effects of utility value and cost reduction interventions in college physics. *Journal of Educational Psychology*, *112*(1), 166–182. https://doi.org/10.1037/edu0000370

Seah, W. T. (2005). *The negotiation of perceived value differences by immigrant teachers of mathematics in Australia*. [Doctoral thesis]. Monash University. https://doi.org/10.4425/03/59c9f2bfd2481.

Seah, W. T. (2019). Values in mathematics education: Its conative nature, and how it can be developed. *Research in Mathematics Education, 22*(2), 99–121. https://doi.org/10.7468/jksmed.2019.22.2.9

Seah, W. T., & Andersson, A. (2015). Valuing diversity in mathematics pedagogy through the volitional nature and alignment of values. In A. Bishop, H. Tan, & T. N. Barkatsas (Eds.), *Diversity in mathematics education: Towards inclusive practices* (pp. 167–183). Springer International Publishing. https://doi.org/10.1007/978-3-319-05978-5_10

Stephens, M. (2019). Embedding algorithmic thinking more clearly in the mathematics curriculum. In Y. Shimizu, R. Withal (Eds.), *Proceedings of ICME 24 School mathematics curriculum reforms: challenges, changes and opportunities* (pp. 483–490). University of Tsukuba.

ENACTING MULTIPLE POSITIONS IN BECOMING A MATHEMATICS TEACHER EDUCATOR

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This paper initiates a longitudinal study that explores the growth and development of mathematics teacher educators (MTEs), particularly those transitioning from mathematics teacher to university-based MTE. While existing research often employs self-based methodologies, this study adopts positioning theory as an alternative approach, examining Mikaela, a lower primary school teacher transitioning to a university-based MTE in Sweden. The paper contributes to the broader understanding of growth and development, offering insights into the challenges and strategies involved in transitioning from mathematics teacher to MTE. The study marks the beginning of a more extensive exploration of this transition process, emphasising the need for a nuanced conceptualisations of MTE learning and expertise.

BECOMING A MATHEMATICS TEACHER EDUCATOR

With this paper, we mark the starting point of a longitudinal study concerning the process of becoming mathematics teacher educators (MTEs). We consider MTEs to be those practitioners who are "engaged in the education or development of teachers of mathematics" (Beswick & Goos, 2018, p. 418) and conceptualise 'becoming' as an ongoing process of professional growth and development that implies agency and movement. MTEs as a group include those who work with prospective teachers as part of their initial teacher education (ITE) and those commonly referred to in the literature as "PD facilitators" (e.g., Prediger et al., 2022) who work with in-service teachers as part of their ongoing professional development (PD). Though these groups of MTEs overlap, PD facilitators are often experienced school-based practitioners who take on the role of leading PD programmes for in-service mathematics teachers whilst MTEs who work in ITE settings are more frequently based in universities and teach mathematics or mathematics education/methods courses for prospective mathematics teachers. Our research concerns this latter group of MTEs who work in university-based ITE settings. Though it is in no way universal, it is commonly the case that these MTEs have been mathematics teachers themselves "and bring a profound professional experience to their work with teachers" (Jaworski, 2008, p. 1). For the purposes of our research, we focus on university-based MTEs who have made this transition, a group who have received modest attention in mathematics education research. We see the initial research described in this paper as shaping a more extensive study on the process of becoming MTEs in different regions and contexts, over a period of several years. This is an important line of inquiry that aims to draw on the journeys of multiple individuals to inform a conceptualisation of the multifaceted and complex nature of MTE learning and expertise, going beyond a categorisation of MTE knowledge.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 192–199). PME.

MTE GROWTH AND DEVELOPMENT

Research on MTE growth and development more broadly is beginning to receive increasing attention from mathematics education scholars, especially those whose research relates to the field of mathematics teacher education and who recognise the vital role MTEs play in shaping the learning and practices of mathematics teachers. In relation to university-based MTEs, this research is often enacted by MTE-researchers who utilise self-based methodologies with the purpose of understanding and improving their own teaching practices (Suazo-Flores et al., 2021). These self-based methodologies include *autoethnography* (e.g., Ward, 2017); *narrative inquiry* (e.g., Bailey, 2008); or *self-study* (e.g., Kastberg et al., 2019), where practising MTE-researchers engage in close-to-practice research, either as individuals or as part of a collaborative group. While research on the growth and development of university-based MTEs is becoming more established within mathematics education, published research that utilises alternatives to self-based methodologies (i.e., where the researchers are not the participating research subjects), as in this study, are less common. Examples include Goos and Bennison's (2018a; 2018b) work who report on a collaborative project between MTEs consisting of mathematicians (from university mathematics departments) and mathematics educators (from university education departments) across several Australian universities. The authors interviewed participating MTEs to identify "interdisciplinary boundary practices" (Goos & Bennison, 2018a, p. 255) that led to the co-development of new courses which were co-taught by participating MTEs from both disciplines (mathematics and mathematics education). In a follow on study (Goos & Bennison, 2018b), the same authors analysed the development of MTEs by focussing on interviews with two participants from the previous study (one mathematician and one mathematics educator). Along with providing important findings in relation to establishing effective cross-disciplinary collaborations within mathematics teacher education and insights regarding the growth and development of MTEs, Goos & Bennison also provide examples of approaches to researching university-based MTEs in ITE settings beyond self-based approaches.

We situate our more extensive study within the growing body of research relating to the growth and development of MTEs, and specifically the ongoing process of transitioning from mathematics teacher to university-based MTE (in ITE settings). In relation to this process of transitioning, more research exists outside of mathematics education. Dinkelman et al. (2006), for example, report on the struggles that two beginning teacher educators experienced in becoming teacher educators, a process they describe as *recasting* their teacher identities. Similarly, Amott (2018) discusses the problematic nature of the transformation of school teachers who become teacher educators, a transition that has been described as "expert become novice" (p. 477).

THE CASE OF MIKAELA

Our starting point, and the focus of this paper, is an interview with Mikaela who herself is in the early stages of transitioning from working as a mathematics teacher to working at university as an MTE. We intend to use the findings from this initial study to raise questions that will form the basis of our more extensive study (see 'discussion' section). Mikaela is currently working as a lower primary school teacher which she combines with a part-time (0.2) position at a university in Sweden where she teaches on mathematics education courses for prospective primary teachers. Mikaela is currently a teacher in grade one (students aged 6-7 years) and has several other responsibilities at school. For instance, Mikaela manages a team of teachers (pre-school to grade 3), she is a school-based mentor for prospective teachers completing their practicum, and she is a mentor for early career teachers. She is also a participating teacher in a research project in Sweden called Problem-solving in Preschool class/Problemlösning i Förskoleklass (PiF), and is studying a master's course in pedagogical leadership.

POSITIONING THEORY

We view the process of becoming an MTE as a relational process that builds from the assumption that we know in relation to others and that "knowing through relationship to self and others is central to teaching" (Hollingsworth et al., 1993, p. 8). Through the process of becoming, MTEs continuously position themselves and others in relation to mathematics, mathematics teaching, students, teachers and other MTEs, as well as other aspects of their profession. To focus on the relational aspects that Mikaela brings forth during the course of the interview, we use positioning theory (Harré et al., 2009). Positioning involves individuals situating themselves or being situated by others in different situations (Davies & Harré, 1999). Positioning is dynamic and is influenced by factors such as familiarity with a situation, experience level, or perceived status. In other words, how a person positions themselves evolves and differs based on the specific situation or context. This variability can occur almost simultaneously. Positioning can also be strategic, with individuals telling different stories about themselves depending on how they wish to present themselves. Thus our experiences and histories impact our positioning (Davies & Harré, 1999).

Positioning theory proposes that individuals, by means of their communication, actively influence and are influenced by the social positions they assume across different contexts. Positions encompass the roles and relationships Mikaela establishes that concern others. Through positioning, we can start to appreciate some of the complexity of becoming an MTE by observing how Mikaela positions herself in various relations to different actors and the counterimages she uses when positioning herself in these relations. Harré et al. (2009) claim that positioning theory allows us to look at what a person can and cannot do. We can also begin to interpret which acts Mikaela values over others and how Mikaela wants to position herself as an MTE in the future. Drawing on Harré et al. (2009), we ask the following research questions (RQs):

In what ways does Mikaela position herself and others in the process of becoming an MTE?

How are these positions negotiated and what acts are valued in these positions?

METHODS

The interview with Mikaela was conducted in English by both authors and lasted approximately two hours (split across two days). In the first part of the interview we focussed on Mikaela's experiences as a classroom teacher. During this part of the interview, Mikaela was asked to talk about her past and present mathematics learning and teaching experiences. She also talked about her teaching practice and offered descriptions of herself as a teacher. She was asked to describe herself as a mathematics teacher and later as a mathematics teacher educator. Because Mikaela raised topics relating to her own experiences of teacher education, the second part of the interview concerned her experience as a prospective teacher and how she now views her role as an MTE. A large part of the interview related to Mikaela as an MTE and how this relates to her continuing role as a mathematics teacher. In the final stages of the interview, Mikaela described the various challenges and successes she has experienced in the process of transitioning from being a mathematics teacher to an MTE.

To investigate our RQs, we draw on the work of Wagner and Herbel-Eisenmann (2009) who suggest a series of questions when investigating classroom positioning within mathematics education. We adapt their questions for our own use in investigating positioning in relation to the process of becoming an MTE. Firstly, we asked: *who teaches mathematics/mathematics education to whom?* By doing this we were able to identify, from the interview transcript, the relevant actors brought forth during the interview process. For example, Mikaela as a teacher, teaching students and prospective teachers or other teachers or MTEs teaching mathematics or mathematics education. Secondly, we asked *what processes are these different actors engaged in?* By doing that, we were able to identify the different teaching actions these teachers perform. For example, Mikaela described other teachers' teaching, and by doing that, we can interpret how she values the kind of teaching she describes. Finally we asked: *Who are these actors doing these things for and why?* By doing that, we can discuss what is valued by Mikaela and the different roles available those actors in the interview.

RESULTS

NVivo (software for conducting qualitative data analysis) was used during the process of analysing the interview transcript. Specifically, sections of text relating to specific actors were coded, resulting in five different positions being identified: 1) Mikaela as a teacher (preschool and lower primary) teaching young children and students of mathematics; 2) Other teachers teaching mathematics; 3) Mikaela as an MTE teaching prospective teachers; 4) MTEs teaching Mikaela; 5) Mikaela as a mathematics teacher, participating in an ongoing research project (PiF). In relation to the first position, Mikaela uses the second as a counter position while negotiating her valued acts in the classroom. Mikaela negotiates her position as a teacher by using other actors to promote herself and explain the position she has taken at the university. She uses teachers to position herself as a teacher who is “up to date” with the national curriculum and somebody who is engaged in research-informed teaching. For this reason we combine these two positions in the analysis below.

In relation to the third and fifth positions, Mikaela refers to herself as a learner of mathematics education. As a learner of mathematics education, several actors benefit: her current students, herself, and the prospective teachers. She uses MTEs from the past in contrast to her own positions when negotiating her new role as an MTE. For this reason, we also combine these three positions in the analysis below. The following two sections present a narrative interpretation of Mikaela's positioning with some extract of the interview transcript woven into the text. We address what processes these different actors are engaged in, who these actors are doing these things for, and why [note, [...] indicates a pause].

As a teacher teaching mathematics

As a mathematics teacher, you can do what you want “because no one enters your classroom”. Mikaela considers the high expectations from principals in Sweden as empty words. However, Mikaela has high expectations of herself and wants to feel valued for her work. Mikaela values a problem-solving approach to teaching, and thinks that, as a teacher, you should engage and enact teaching using multiple representations, multimodal teaching and learning mathematics through the arts. She knows this from both research and her own experience. She is giving children ways to become learners of mathematics.

When positioning herself with other teachers, Mikaela highlights her problem-solving teaching ability. She promotes teaching in line with this ability to position herself against the other teachers. This puts her in a position where she teaches correctly. She wants more than “just watching them play” or “keeping on doing the things they did for the last 25 years”. Mikaela values educational reform. When the educational demands are changing, “you must change [...] it is not okay for teachers to continue doing things as they have always been doing”. Mikaela positions these teachers as neglecting students’ right to learn and reach their full potential, “it is their character”. Mikaela emphasises that she has different character; she wants to make a difference. She says, “I am not that kind of person”, and positions herself as a thoughtful teacher caring about her students' learning. For Mikaela the means being thoughtful and aware of how children learn and acting accordingly, which means changing children’s thinking, changing their way of solving problems, and guiding their learning.

From a participant in PiF to becoming an MTE

PiF was the starting point for Mikaela’s desire to become an MTE. When she got the opportunity to teach prospective teachers she found it rewarding; the positive response from the prospective teachers had a long-lasting impact on her. Mikaela positions herself as having a real impact on the prospective teachers and that both her teaching experience and her teaching are valued as being “up-to-date”. In educating prospective teachers, Mikaela feels she can foster discussion; “everyone is relaxed and allowed to speak". PiF has contributed to her development as a group leader, and she has "developed a leadership personality" by facilitating a problem-solving classroom in PiF that she now enacts as an MTE. She does not view herself as teaching differently as an MTE when comparing that to her teaching in schools; "teaching is teaching, there is no difference, only different content". For Mikaela, prospective teachers value MTEs who are engaged, interested and settled in the teaching profession. This supports them in developing an ability to relate to practice. PiF has also given her a teaching context and content to elaborate on when she teaches prospective teachers. She positions herself as a facilitator, and by doing that, she is a competent MTE with inclusive and respectful communication with those she teaches.

The main difference between being a mathematics teacher and an MTE is the level of demand that she feels from the prospective teachers and her new colleagues at the university. She values these demands, but it is "a little uncomfortable". Mikaela concludes, "as a teacher educator, you have a lot of responsibility and high expectations from others". Mikaela positions herself again in relation to others, this time other MTEs. Her teaching applies to today's school system, and prospective teachers need proper teaching because they may end up being future colleagues a few years from now. She bridges two practices; she gives practical examples of theoretical ideas as a way of linking theory and practice, even when the content and context are new to her:

> I must read the literature very closely and understand it myself before teaching it. That has been in areas where I have no experience, like Variation Theory. I was new to that and became a teacher educator in that area. I had to educate myself first to feel comfortable myself. However, when I had done that, I could see how my teaching was connected with it. I could give students practical examples of how they could work. However, I had not thought about it before then.

Mikaela is aware of all the different positions she has at the moment. She concludes the interview by saying:

> It will be fun when I, as a teacher, get one of my students from the university on their internship, and then one of my colleagues from the university comes to visit. Because then they have been students at the university and have seen me both in the classroom with children and as a teacher educator. Maybe I can get some feedback on my different roles.

DISCUSSION

We now look forward to following Mikaela and the other MTEs through their journeys in becoming MTEs. At the time of the interview, Mikaela seems to be acting out multiple positions simultaneously (Davies & Harré, 1999) as she transitions from being a mathematics teacher in school to becoming a university-based MTE. Moving forward we ask ourselves whether Mikaela's positioning will evolve and in what ways.

In this paper, we have explored Mikaela's current positioning as a starting point from which to study her process of becoming an MTE as an ongoing process of professional growth and development. We view the use of positioning theory (Davies & Harré,

1999) as an analytical framework to enrich our understanding of the relational dynamics involved in this transition. By departing from self-based methodologies, we argue that this paper and study contribute a promising perspective to understanding the learning and expertise of university-based MTEs in ITE settings.

In Mikaela's roles as a mathematics teacher, an MTE teaching prospective teachers, and a participant in a research project, she strategically emphasises the teaching of problem-solving and promotes educational reform through her positioning. By focusing on relational aspects of becoming an MTE we see ways in which *"I"* is positioned in relation to the *"other"*. How will the counterimages used by Mikaela change over time? The results of this study begin to reveal the nuanced shifts in responsibilities and expectations, which seem important to Mikeala as a beginning MTE. Mikaela's journey from a participant in PiF to teaching prospective teachers at university, underscores the challenges associated with this transition, from our point of view, offering valuable insights into the complexities of becoming an MTE. As Mikaela reflects on her various roles, it opens up avenues for her to explore her own process of transitioning and the impact that former MTEs have on her unfolding story. Her self-awareness and anticipation of feedback add another layer to the study. How is being affirmed guiding her in future positions?

As intended, this initial study raises questions that will form the basis of our more extensive study, for instance, what is the role of personal gain and self-affirmation in the process of transitioning from classroom teacher to university-based MTE? How is this personal gain reflected in our actions in mathematics teacher education and how does this change over time? How is past teaching experience transformed in the process of becoming an MTE? In summary, this paper contributes to the research on MTE growth and development by examining the relational aspects of transitioning from mathematics teacher to university-based MTE. Mikaela's case serves as a compelling starting point, paving the way for further inquiries into the multifaceted nature of MTE learning and expertise in diverse educational settings.

REFERENCES

Amott, P. (2018). Identification – a process of self-knowing realised within narrative practices for teacher educators during times of transition. *Professional Development in Education, 44*(4), 476–491.

Bailey, J. (2008). Using narrative inquiry to explore mathematics curriculum. *Curriculum Matters, 4*, 147–163.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education, 21*, 417–427.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics, 93*, 19–33.

Davies, B., & Harré, R. (1999). Positioning and personhood. In R. Harré, & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 32–51). Blackwell

Dinkelman, T., Margolis, J., & Sikkenga, K. (2006). From teacher to teacher educator: Experiences, expectations, and expatriation. *Studying Teacher Education, 2*(1), 5–23.

Gee, J. P. (2004). *Situated language and learning.* Routledge.

Goos, M., & Bennison, A. (2018a). Boundary crossing and brokering between disciplines in pre-service mathematics teacher education. *Mathematics Education Research Journal, 30*(3), 255–275.

Goos, M., & Bennison, A. (2018b) A zone theory approach to analysing identity formation in mathematics education. *ZDM Mathematics Education, 51*(3), 405–418.

Harré, R., Moghaddam, F. M., Cairnie, T. P., Rothbart, D., & Sabat, S. R. (2009). Recent advances in positioning theory. *Theory & Psychology, 19*(1), 5–31.

Holland, D., Skinner, D., Lachicotte, W., & Cain, C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.

Hollingsworth, S., Dybdahl, M., & Minarik, L. (1993). By chart and chance and passion: The importance of relational knowing in learning to teach. *Curriculum Inquiry, 23*(1), 5–35.

Jaworski, B. (2008). Mathematics teacher educator learning and development. In B. Jaworski, & T. Wood (Eds.), *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 1–13). Sense.

Kastberg, S., Lischka, A., & Hillman, S. (2019). Exploring mathematics teacher educator questioning as a relational practice: Acknowledging imbalances. *Studying Teacher Education, 15*(1), 67–81.

Lloyd, M. (2022). Teacher educators' general beliefs and personal identifications related to mathematics. *Mathematics Education Research Journal.*

Potter, J., & Wetherell, M. (1987). *Discourse and social psychology. Beyond attitudes and behaviour.* Sage.

Prediger, S., Roesken-Winter, B., Stahnke, R., & Pöhler, B. (2022). Conceptualizing content-related PD facilitator expertise. *Journal of Mathematics Teacher Education, 25*(4), 403–428.

Suazo-Flores, E., Kastberg, S., Grant, M., Ward. J., Richardson, S. E., & Chapman, O. (2021). Using self-based methodologies to unpack mathematics teacher educators' work. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proc. 43rd Conf. of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 1907–1910). PME NA.

Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention. *Educational Studies in Mathematics, 72*(1), 1–15.

Ward, J. (2017). *Early childhood mathematics through a social justice lens: An autoethnography*. [Doctoral thesis, University of South Florida]

VICARIOUS LEARNING SCRIPTED VERSUS UNSCRIPTED VIDEOS: PROBLEM-SOLVING BEHAVIORS

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Vicarious learning research is a growing area of inquiry examining the learning of students who observe video-/audio-taped students engaged in learning (Mayes, 2015). To date, several projects have reported on the learning gains of indirectly participating in dialogue. However, an important question remains about the influence the nature of the dialogue—whether it is scripted or unscripted—has on viewers. For this study, two sets of dialogic videos were created capturing the inquiry process of students engaged with either unscripted or scripted dialogue. Each video type was shown to a pair of students over five research sessions. Using thematic analysis, patterns and differences between how each pair used their respective set of videos were identified. Preliminary findings suggest a difference in the pairs' problem-solving behaviors.

INTRODUCTION

The use of instructional videos has grown rapidly, particularly videos covering mathematical content. This is evidenced by the breadth and popularity of instructional videos available online for mathematics education (e.g., Khan Academy) and the growing use of instructional videos in educational settings (e.g., flipped classrooms). As a result, there is a need to understand how instructional videos should be formatted to best serve our students and how videos are being used by students to learn. One promising style of instructional videos stems from the literature positioning its viewers as *vicarious learners* (VLs), or indirect participants in another's learning process.

The goal of this study was to identify what problem-solving behaviors emerged from VLs as they engaged with the mathematics within a series of dialogic videos centered on the construction of the sine function and to determine if the VLs' problem-solving behaviors differed when the videos they viewed contained scripted versus unscripted dialogue. Following Schoenfeld (2016), problem-solving behaviors includes the cognitive processes or strategies used to identify, consider, and solve novel problems. During the research sessions, the VLs were given novel problems and videos containing students (i.e., the talent) working on the same problems. One set of videos contained the unscripted dialogue of the talent's progression toward the construction of the sine function. A second set of videos were filmed following a scripted dialogue written to reproduce the unscripted dialogic videos using the solutions and emergent difficulties from the unscripted dialogue. While working on the novel problems, the VLs had autonomy in their use of their videos, and they could choose to watch the videos and work on their tasks as they saw fit. As such, viewing and engaging with the dialogic videos were integral to the VLs' problem-solving behaviors.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 200–207). PME.

LITERATURE REVIEW

Following Mayes (2015), vicarious learning is defined as the indirect participation in a process of learning mediated by technology. A VL is then the individual that is indirectly participating in learning. According to McKendree et al. (1998), one of the affordances of indirect participation is VLs' opportunity to listen in on and observe explicit connections made within a dialogue. Observing dialogue allows students to see connections others are making, ones they may have missed, and the process of making connections, serving as a model for learning for the VLs.

A substantial portion of the vicarious learning literature has focused on comparing the learning gains on pre/post-test measures from vicarious learning treatments who engaged with dialogic material to treatments who used monologic material (e.g., Muller et al., 2007). For example, Muller et al. (2007) conducted a pre/post-test comparison of students who viewed scripted monologic videos versus scripted dialogic videos. In constructing the two forms of videos, both videos covered the same content. The difference between the videos was the presence of scripted questions and alternative conceptions presented by a tutee and resolved by a tutor in the dialogic videos. In their study, Muller reported a statistically significant difference in the learning outcomes in favor of the students who viewed the dialogic videos.

In addition to improved learning outcomes of VLs, indirect participation has also been found to produce modeling of observed behaviors (e.g., Gruver et al., 2022). In their study, Gruver and colleagues reported on two ways in which their VLs took ideas contained within a series of unscripted dialogic videos and made use of them for their work, a process Gruver's team called ventrilloquation.

Across the growing body of literature on vicarious learning, two gaps have been identified. Foremost, an emphasis has been placed on comparisons of monologic versus dialogic videos (Gruver et al., 2022). Within these studies, the nature of the dialogue has varied (i.e., some studies implemented scripted dialogue and others have used unscripted). What is missing, then, is a comparison of how the nature of the dialogue influences VLs' experiences. Additionally, inquiry into what students do while watching instructional videos and how the videos are being used to learn is an understudied area (Weinburg & Thomas, 2018).

METHODS

Participants and Research Context

Over five research sessions, two pairs of VLs viewed dialogic instructional videos that contained unscripted or scripted dialogue. In total, four participants were recruited for this study. Each participant was assigned a partner, creating two pairs. Each pair was then assigned a video type, scripted or unscripted. The pair of students watching the unscripted videos (henceforth referred to as the Unscripted VLs, U-VLs) were Sarah and Osiris and the pair watching the scripted videos (henceforth referred to as the Scripted VLs, S-VLs) were Camila and Alex (pseudonyms used).

The mathematics contained within both sets of dialogic videos centers on a sequence of tasks where the students are asked to consider a point moving around a shape. Within this context, the problem the students are solving is to construct a graph that relates the distance traveled by the point to the height of the point. The tasks provide a sequence of *n*-sided polygons whose graph approaches the sine function as *n* increases. Fundamental to the task progression is the integration of rich covariational reasoning that incrementally increases in complexity for each subsequent task. This sequence culminates in the development of covariational reasoning for the resultant function.

To illustrate the VLs' problem-solving behaviors, their work on Square Task 1 and the Octagon Task (Figure 1) was analyzed. These tasks were identified because they captured the treatments' work on the same task, the pairs have a similar starting and ending position for both tasks, and the episodes are characteristic of both pairs' behaviors across all their respective research sessions. Furthermore, these tasks mark significant milestones in students' covariational reasoning toward their construction of the sine function. For example, Square Task 1 requires the students to coordinate three different changes in height (i.e., height increasing, decreasing, or constant) with a continuous increase in distance traveled.

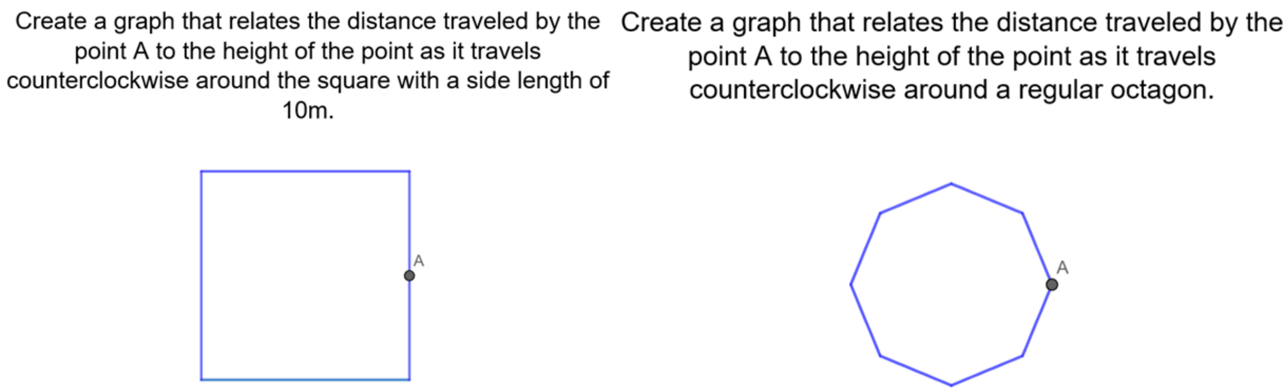


Figure 1: Square Task 1 and the Octagon Task.

Thematic Analysis

Thematic analysis (TA), as systematized by Braun and Clarke (2006), was used to identify and describe patterns of meanings or themes within the VLs' problem solving. Braun and Clarke delineated six phases within TA. Phase 1 begins with the familiarization of oneself with the data. Phase 2 calls for the generation of initial codes. Phase 3 entails the creation of general themes and subthemes. Phase 4 involves reviewing and clarifying themes created in Phase 3. Phase 5 requires defining and naming themes. Phase 6 involves the production of the research report. Through this iterative process, three themes were identified related to the VLs' use of the videos as a tool for their problem solving: (a) patterns of use, (b) idea justification, and (c) idea management. Exploration of all three of these themes, and their constitutive subthemes, is beyond the scope of this report. As such, we focus on the theme of idea justification.

RESULTS

The theme of idea justification captured the VLs' problem-solving behavior of providing support for mathematical claims while working on novel tasks and engaging with dialogic videos. The Common Core State Standards for Mathematics (2010) state, "One hallmark of mathematical understanding is the ability to justify … why a particular mathematical statement is true or where a mathematical rule comes from" (p. 4). This suggests that the ability to justify a solution is an important mathematical activity. Thus, when justifications were presented by the VLs, insight into their mathematical understandings (e.g., their covariational reasoning) were identifiable. Furthermore, given the expectation that the VLs' would explain their work to the researcher, idea justification emerged as a part of their expected problem-solving behavior. In total, two sub-themes of idea justification were identified: (a) appeal to an authority and (b) substantive justification.

Appeal to an Authority

Appeals to an authority occurred when a claim was supported by an external source. In their work on proof schemes, Sowder and Harel (1998) identified appeals to an authority as a part of externally based proof schemes. A proof that relied on an appeal to an authority was identified when support for a claim relied on another person, typically a teacher or another source of knowledge (e.g., a textbook). During the research sessions, only the S-VLs offered an appeal to an authority as a justification.

One of Camila's and Alex's appeals to an authority occurred during their work on Square Task 1 when they appealed to the authority of the talent. Before their work on the task, the S-VLs watched the accompanying video, and after watching the talent complete the task, Camilia and Alex worked on the task. During their work, Camila claimed that the starting height of point *A* was 0 meters. Her justification for this claim came at the end of a lengthy back-and-forth, that included an intermediary justification for the chosen starting height based on the presence of negative values. Then Camila's justification for this intermediary claim relied on an appeal to the authority of the talent.

After Camila completed her graph (Figure 2), Alex claimed that Camila's graph was wrong, despite seeing the talent make the same graph, because "a graph doesn't always have to start at 0." Camila then attempted, but struggled, to justify her starting location:

Alex: Would it be 0 right here [points to the point opposite point *A* on the left side of the square, Figure 2]?

Camila: **That's also 0 because we have this** [points at -5 on *y* axis]. So the -5 would be down here [gestures from the midpoint on the left side of the square to the bottom left corner labeled -5 in Figure 2].

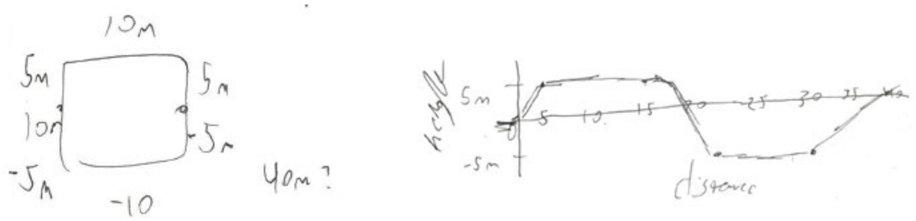


Figure 2: Camila's square and graph for Square Task 1.

Within this exchange, Alex attempted to explore the implication of calling the starting height of point *A* 0 meters. In response, Camila claimed that "[the height is] 0 because we have [negative values]." Here, Camila has suggested that the presence of the negatives is a justification for the starting height being 0. In response, the researcher asked the pair a clarifying question about what the VLs thought was positive and what was negative. In response, Camila clarified what was negative and appealed to the authority of the videos as justification for her negative values:

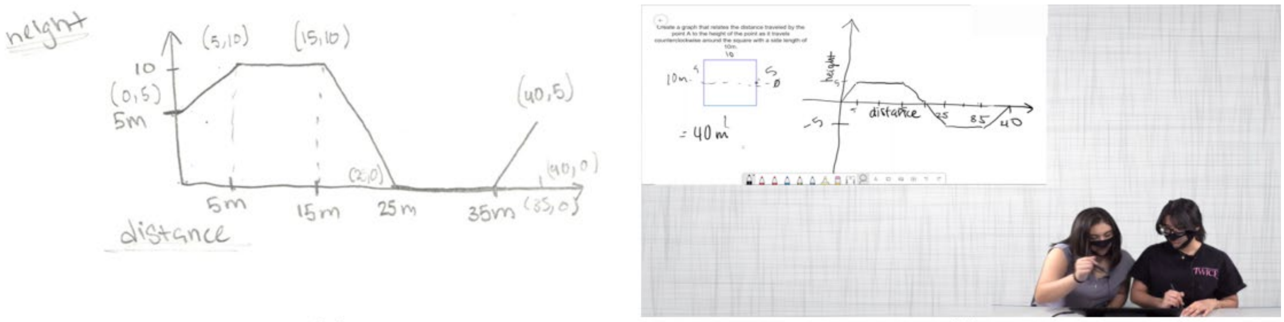
Researcher:	What's negative and what's positive here?
Camila:	The negative is the down, and the positive is the up.
Alex:	Below this line [traces a midline across the square in Figure 2 with his finger] it's negative.
Researcher:	What's negative?
Camila:	The number. The height. Because **he [the talent] said it was** [points at laptop]…

In clarifying what the VLs were discussing, both Alex and Camila indicated the bottom half of the square represented negative values. Furthermore, Camila clarified that the negative quantity was the height. She then justified this claim by appealing to the authority of the talent when she said, "[the talent] said it was." In sum, Camila first justified her claim that the starting height of point *A* is 0 meters by appealing to the presence of negative values on her square. Then, she justified her negative values by appealing to the authority of the talent. Thus, Camila's justification for her claim that the starting height of point *A* is 0 meters was an appeal to the authority of the talent.

Substantive Justification

A substantive justification is one that included support for a claim ranging from the use of an example, a fact, or deductive reasoning. This form of justification captured the VLs' attempts to create a justification that leveraged mathematical facts or concepts in support of their claims. Importantly, the VLs' substantive justifications were not always mathematically correct, and their justifications may not have validity in all mathematical contexts. Instead, these justifications captured the VLs' attempts to create arguments centered on their current mathematical understandings.

For example, the U-VLs, Sarah and Osiris, leveraged a substantive justification in support of the claim that the starting height for point *A* on Square Task 1 is arbitrary and that it is acceptable to have negative heights. While working on Square Task 1, Sarah and Osiris produced a graph (Figure 3a) that differed from the graph produced by the talent in the video (Figure 3b).



(a) (b)

Figure 3: (a) the U-VLs graph for Square Task 1 and (b) the graph produced in the unscripted videos for Square Task 1.

Initially, the VLs struggled with the difference between the graphs and argued that the talent were wrong because Sarah and Osiris believed it was incorrect to claim the starting height of point *A* was 0 meters. Furthermore, Sarah claimed, “I don’t think you can really go negative,” suggesting she did not believe you can have negative heights. Osiris disagreed with Sarah and was able to construct a substantive justification for his idea that the talent’s starting height of 0 meters and the presence of negative values are valid. Osiris stated,

> Actually, I think there’s no actual reason, I don’t think it matters that much. I don’t think it matters at all. **If you were to assume this** [points to the starting point on Square Task 1] **was 0**, imagine this was 0. **This would be 0 on this side too** [points to midpoint on left side of Square Task 1]. So, if they go up 5 [sweeps from the starting point to top right corner], go here 10 [sweeps across top of the square] go down 5 [points to the midpoint on the left side the square] it would be 0 still. Go down -5, **which is not wrong if they’re measuring by that** [points to starting point] **being 0**. Because **there is no exact point that says point *A* has a starting point of 5** [points at task statement].

In this excerpt, Osiris claimed that the starting height of point *A* was arbitrary for Square Task 1 and that assuming the starting height was 0 implies the use of negative heights. Osiris substantively justified his first claim by pointing out that the problem statement did not provide an initial “starting point of 5” or 0. In fact, Osiris pointed out that this is just an assumption. Finally, he provided a substantive justification for the presence of negative heights by describing the point’s journey around the square, based on the assumed starting height of 0 meters. This description leveraged his covariational reasoning to describe the simultaneous change in distance with the change in height of the traveling point.

CONCLUSION

Justifications are an important part of problem solving, and the different ways in which the VLs attempted to create justifications for their claims have implications about their mathematical understandings of those claims. When the U-VLs constructed justifications, it was substantive. This means that the claims they made were rooted in their mathematical understanding (i.e., supported by examples, facts, or deductive reasoning). This was evidenced by Osiris's ability to reason covariationally about both his and the talent's graph. Although the S-VLs produced similar substantive justifications, they also produced justifications rooted in appeals to an authority. This latter type of justification was not grounded in their mathematical understanding of covariational reasoning, and instead evidenced Camila's struggle to defend her graph.

To explore the difference between the U-VLs and the S-VLs, thematic analysis was used to identify themes in the VLs' uses of the videos to complete Square Task 1 and the Octagon Task. While an exploration of all identified themes was beyond the scope of this report, each of the three identified themes (patterns of video use, idea justification, and idea management) indicated important features of the VLs' problem-solving behaviors and differences between the treatments' problem-solving behaviors.

The U-VLs' problem-solving behaviors appeared to be driven by the goal of constructing a solution, and the S-VLs' problem-solving behaviors appeared to be driven by the goal of having a solution. Foremost, the U-VLs' process of constructing a solution was evidenced in their problem-solving behavior of creating substantive justifications and their lack of appeals to an authority. Through the creation of substantive justifications, Sarah and Osiris demonstrated an understanding of their work. Beyond constructing a solution that was true because the videos said it was (i.e., an appeal to an authority), the solutions that Sarah and Osiris found could be supported mathematically. Additionally, the U-VLs' patterns of use and their idea management reinforced their constructive process. The S-VLs, on the other hand, appeared to focus their problem-solving behaviors on possessing a solution. This was evidenced in Camila's and Alex's patterns of use, the justifications they created, and the way they managed ideas from the videos. During the justifications the pair created, this was demonstrated through their appeal to an authority. This form of justification showed Camila and Alex had a lack of understanding of the solutions they had produced and were content to have an answer they knew was write because it aligned with the video.

We end with a consideration of the limitations of this work. As previously mentioned, the scope of this research report narrows the reporting of our findings to one of our identified themes. Additionally, we are limited by our sample size. While we begin to explore and identify a set of problem-solving behaviors, our claims cannot extend beyond the behaviors of our VLs and the difference in their uses of the scripted and unscripted videos. Finally, this study is limited in its ability to connect the identified differences, found through a qualitative analysis, to measures of learning outcomes. Future work will fully present our identified themes, and explore the connection between learning outcomes and the identified problem-solving behaviors.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101.

Gruver, J., Lobato, J., & Foster, M. (2022, July). Investigating the learning process of students using dialogic instructional videos. *Proc. 45th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol 2., pp. 323-330). Alicante, Spain.

Mayes, J. (2015). Still to learn from vicarious learning. *E-Learning and Digital Media, 12*(3–4), 361–371.

McKendree, J., Stenning, K., Mayes, T., Lee, J., & Cox, R. (1998). Why observing a dialogue may benefit learning. *Journal of Computer Assisted Learning, 14*(2), 110-119.

Muller, D. A., Sharma, M. D., Eklund, J., & Reimann, P. (2007). Conceptual change through vicarious learning in an authentic physics setting. *Instructional Science, 35*(6), 519-533.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards Mathematics*. Washington D.C.: Author. http://corestandards.org/

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education, 196*(2), 1-38.

Sowder, L., & Harel, G. (1998). Types of students' justifications. *The Mathematics Teacher, 91*(8), 670-675.

Weinberg, A., & Thomas, M. (2018). Student learning and sense-making from video lectures. *International Journal of Mathematical Education in Science and Technology*, *49*(6), 922-943. https://doi.org/10.1080/0020739x.2018.1426794

USING TASK VARIATION TO SUPPORT ARGUMENTATION IN COLLABORATIVE PROBLEM SOLVING

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There is a need for more research on how to support argumentation in mathematics classrooms. This study describes how task variation can be used to support argumentation in mathematics classrooms. The study uses data from an NSF-funded longitudinal study on how students reason while working collaboratively on challenging mathematical tasks. Findings show that variation of the structure and implementation of tasks can help students build valid arguments.

INTRODUCTION

Argumentation is an important disciplinary practice that should be promoted in mathematics classrooms. The Principles and Standards for School Mathematics (NCTM, 2000) emphasize reasoning and proof as well as communication, three essential components of argumentation. The Common Core State Standards for School Mathematics (CCSSM, 2010) state that students should be able to "Construct viable arguments and critique the reasoning of others" (p.7). The CCSSM further assert that being able to justify or derive mathematical statements, essential in mathematical argumentation, is an indication of mathematical understanding. However, research shows that teachers find it challenging to support argumentation in mathematics classrooms and more research is needed on teacher knowledge and practice of argumentation. This study shows that task variation can be used to support argumentation in classrooms.

THEORETICAL BACKGROUND

Argumentation and teaching

According to Toulmin's scheme of argumentation, an argument consists of three essential parts. A *claim* is the assertion of which an individual is trying to convince others. The *data* are the evidence that the individual presents to support the claim. The *warrant* is the explanation of why the claim follows from the data. Members of a group may not be convinced that a claim follows from the data and may question the validity of the warrant and the individual may present a support or *backing* for the warrant. The scheme has two additional components: a modal qualifier, which refers to the degree of confidence about a claim, and a rebuttal, which refers to the conditions under which conclusions may or may not hold. These components provide a more comprehensive description of individuals' argumentation and reasoning processes and help investigate arguments similar to those made by mathematicians (Inglis et al. 2007). However, there is a general consensus that arguments produced by students in schools are different

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 208–215). PME.

from arguments of advanced mathematics students and it is not necessary to use the full scheme to analyze the arguments (Knipping and Reid, 2019, p.5).

Krummheuer extended Toulmin's notion of argumentation from an individual to a collective notion distinguishing between situations where one individual tries to convince an audience about the validity of a claim and situations where two or more individuals interact to establish a claim, which he called *collective argumentation*. This makes collective argumentation an interactional accomplishment. An argument can no longer be analyzed only by considering a sequence of statements that are made. The functions that various statements serve in the interaction of participating individuals become critical to making sense of argumentation that develops. What constitutes data, warrants, and backing is no longer predetermined, but negotiated by participants in interaction. Collective argumentation is also a useful construct for analysing teacher's role in supporting argumentation in interaction with students (Yackel, 2002).

Teachers play a key role in supporting argumentation. They can negotiate classroom norms that foster argumentation, support students as they interact with each other to develop valid arguments, and supply argumentative supports (data, warrants, and backing) that are omitted or left implicit in students' arguments (Yackel, 2002). They can implement tasks that support conceptual understanding and learning about argumentation (Kosko et al., 2014), prompt students to critically consider arguments, present what constitutes acceptable arguments, and model ways of constructing and confronting arguments (Ayalon & Hershkowitz, 2017). However, teachers find it challenging to incorporate argumentation into classroom practice (Conner et al, 2014) and often have interpretations of facilitating argumentation that are not aligned with what reformers in mathematics education envision such as thinking that argumentation can occur with relatively little scaffolding (Kosko et al., 2014) or only the smartest students can engage in rigorous justification (Bieda, 2010). This suggests that more research is needed on teacher knowledge and practice of argumentation.

Tasks and support of argumentation

Tasks play an important role in mathematical learning and teaching. The quality of instruction depends on whether "teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks (Kilpatrick, 2001, p.9)". Challenging tasks help promote productive struggle which is an essential component for learning mathematics with understanding (Hiebert et al, 2007). Zasvlasky (2005) studied the use of "uncertainty-evoking tasks" to support learning of both students and mathematics educators. She noted the dual nature of such tasks in "facilitating both mathematical and pedagogical understandings." She distinguished among different kinds of uncertainty and showed that it is dynamic, subjective, and can stem from tasks as well as social interaction over tasks. She argued that using uncertainty-evoking tasks to support learning is consistent with Dewey's (1933) notion of *reflective thinking*, which entails both "(1) a *state of doubt*, hesitation, perplexity, mental difficulty in which thinking originates, and (2) an act of searching, hunting, inquiring, to find material that will *resolve the doubt*, settle and dispose of the perplexity" (p. 12). This suggests that such tasks can support argumentation by promoting reflective thinking about argumentation, creating doubts in students' arguments and/or prompting them to resolve the doubts and improve their arguments.

Variation theory

Variation theory focuses on *explaining why* variation in experience exist and using the knowledge to improve teaching and learning. In variation theory learning is viewed as a change or expansion of a learner's structure of awareness of a phenomenon as the learner comes to see the phenomenon in new, more complex ways deemed more appropriate by the teacher (Orgill, 2012). This requires simultaneously discerning *critical features* or aspects of a phenomenon that help the learner see the phenomenon in a particular way. However, discerning a feature requires experiencing variation in dimensions of the feature. This makes experiencing variation a necessary though not sufficient condition for learning. It makes learning possible, but does not guarantee it.

In Variation theory what is to be learned by students is referred as the object of learning and is examined from three different perspectives. The *intended object of learning* is what instructors think students should learn about the object of learning and it answers to the question "What was intended to be learned?" The *enacted object of learning* is what is possible for students to learn about the object of learning based on how teachers structures learning experiences and it is answer to the question "what is made possible to learn in this classroom?" The *lived object of learning* is what students actually learn about the object of learning and answers the question, "What was learned?"

Variation theory distinguishes among four different patterns of variation (Marton 2015). *Contras*t involves comparing an object of learning with something that it is not. *Generalization* involves comparing similar instances of the same object of learning. *Separation* involves varying only the feature of interest while holding all other features constant or invariant. *Fusion* is variation in which several features of an object of learning vary simultaneously. Different patterns of variation result in different types of learning and variation theory has been used to study how systematic variation can support learning in mathematics. Findings show that "a wisely planned variation, for instance in a *task* or set of examples, can make certain aspects noticeable for the learner (Watson and Mason 2006, p. 109). The research question can be cast as follows: *How can task variation help students build mathematical arguments in collaborative problem solving?*

METHOD

Research context

This study used data from an after-school, classroom-based longitudinal research project on how students reason while working collaboratively on challenging mathematical tasks selected from several content strands. During research sessions students encouraged to always justify their solutions based on whether they were convinced that they made sense. Follow-up interviews helped gain an in-depth understanding of students' reasoning. Debriefing meetings were held at the end of research sessions to discuss how to support students' reasoning. Data for this study came from videos of research sessions involving the following tasks:

The Tower Problem: You have two colors of unifix cubes available to build towers. Make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers and that you have no duplicates.

The World Series Problem: In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won in: a) four games? b) five games? c) six games? d) seven games?

Analysis – Analysis followed Powell et al's (2003) video methodology. Iterations of three sequential and interrelated steps were involved: (1) viewing selected videos of research sessions several times to have a sense of the data as a whole; (2) parsing the videos into episodes of argumentation in which students were building arguments to support their reasoning, and (3) analyzing the episodes for how task variation helped improve arguments. This involved (3.1) coding for elements of arguments (data, claim, warrant, backing) that students were trying to build (Intended object of learning), (3.2) describing task variation used to help improve arguments (enacted object of learning) and, (3.3) examine if it actually helped improve argument (Lived object of learning).

RESULTS

Build claims and warrants

Students tried to solve the World Series Problem by listing game combinations (e.g. AAAA, ABAAA, and ABABAA), but quickly found it challenging to list combinations for series ending in a higher number of games. Jeff suggested determining game combinations as "two to the seventh", but the students were not sure whether to use "two to the *n*th power" or "two to the seventh," and why method was valid. Ankur said for a five-games series, they could use "two to the fifth" to determine all possible game combinations but would have to list winning game combinations:

Jeff: Before we do that. How do you get to that point in the first place? Cause there's a lot of combinations – Is that two to the seventh?

Romina: Isn't it–yeah, two to the n?

Jeff: Yeah. All right, so say it's two to the seventh.

Ankur: For this you gotta find all possibilities. They have eight ways of winning but *it'd be over the total possibilities of two, like two colors and five things*.

Analysis. The students needed to justify using either "to the fifth" or "two to the seventh" to determine game combinations (intended object of learning). Ankur suggests using “two to the fifth” for a five-game series. The expression “*like two colors and five things*” suggests that he was using his previous experience with the Tower Problem in which they came up with “two to the *n*th power” as the general solution. The Tower Problem and the problem of finding all possible game combinations for the World Series Problem introduce variation changing or varying the context of the problem but keeping a similar or invariant (exponential) mathematical structure. This opened the possibility of students seeing the similarity or isomorphism and using it to justify using the same method on both tasks (enacted object of learning). Ankur noticed similarity and used it to justify using “two to the fifth” to calculate all possible game combinations in a five-game series (lived object of learning). Task variation helped build a valid warrant (isomorphism) to support a claim (“two to the fifth”).

Challenge data and reduce uncertainty

In the previous episode, students eventually and came up with the solution: $p(4) = 2/16$, $p(5) = 8/32$, $p(6) = 20/64$, and $p(7) = 40/128$. They said they were confident in their solution but admitted that they were not sure they had listed all winning game combinations. Researchers asked the students to compare their solution to an alternative “solution” proposed by another group of students: $p(4) = 2/70$, $p(5) = 8/70$, $p(6) = 20/70$, and $p(7) = 40/70$. The students noticed the “seventy” but quickly figured out that it represented the sum of all ways of winning the series. They also noticed that in the alternative “solution” p(6) and p(7) were different which they liked since they believed it was easier to win in seven than six games. However, all students except Mike insisted they liked their solution better even though they could not say why the alternative solution was not valid. Mike insisted that he liked in the alternative “solution” better and said “that’s just my opinion.” The other students tried to convince Mike to continue believe in their initial solution but were not successful. The researcher pointed at $p(7) = 40/70$ and asked what it was a probability of. The students said that the probability made no sense because no sample space had seventy games. Ankur suggested that the alternative solution answered a different question. Mike disagreed saying “Ours is wrong’ and “We answered the same question.” The researcher asked if the seventy games were equally likely. The students immediately said they were not equally likely. Ankur said “if it’s an equal chance to win, winning four game in arrow is much harder than winning four games and losing three.” Jeff added “Because that gives you a lot more room for error ‘cause you could slip up there.” Mike started to change his position:

Researcher: Michael, you’re nodding. what do you think?

Michael: I don’t know which one [is right] but the thing where, like, each one of those seventy’s has a different probability if coming up and that’s – if that was the case- I don’t know, but if, if that—I don’t know. If that was the case, then, yes, their [alternative solution] probability is faulty.

Analysis. The students believed in their solution but needed an argument to support its validity (intended object of learning). The argument in support of their solution can be modelled using the completed Toulmin's scheme with a qualifier indicating their high level of confidence in the solution. Asking the students to compare the two solutions introduced task variation keeping the context of the problem invariant but varying the approach as reflected in two solutions. This problematized the sample space of World Series Problem ("two to the *n*th" vs "seventy") opening the possibility of students realizing that one solution could not be valid because the sample space was not valid (Enacted object of learning). The students realized the "seventy" was not a valid sample space and rejected the alternative solution (lived object of learning). The "seventy" as a value of a sample space can be coded as data in an argument supporting the alternative solution (claim). Task variation helped challenge data and reject the claim. Rejecting the alternative solution did not establish the validity of the students' solution, but helped reduce the uncertainty about its validity by comparison to an incorrect solution.

Build warrants and eliminate uncertainty

One reason Mike was not sure about the students' solution was that it did not explain why p(6) and p(7) were the same, which he found counter-intuitive. Researchers organized a separate session in which Mike revisited the World Series problem and worked with another student. Mike finally explained why the probabilities were the same and when asked which solution he now believed in he said the initial solution:

Mike: You know how it doubles from 20 to 40? I was thinking when you have six and if you didn't win at six, what you're going to have is three and three. You're going to have three wins and three losses, whichever way they are.

Researcher: Got you. Ties.

Mike: Yeah, ties, 20 ties. And when you go another game, it can either be a win for one team or a win for the other. So, that's why it would be like—

Researcher: The tiebreaker.

Mike: That's a tiebreaker. You would either have 20 different ways that A would win or 20 different ways that B would win. That's probably why the probability of winning in six is the same, being a tie in six, and when you go another game, it just doubles.

Analysis. Mike was looking for a warrant to justify why the probabilities of the series ending in six and seven games were the same (Object of learning). Variation was in the implementation of the task as Mike worked with another student who brought in a different way of thinking and helped problematize several aspects of the problem showing that uncertainty can stem from tasks as well as social interaction over tasks (Zasvlasky, 2005) and a more expansive view of a task can include how it is implemented.

Empirically challenge warrants

In a session on the Tower Problem, Gabe built four groups of four towers (Figure 1) and said that there are sixteen towers in total "because 4x4 is sixteen." Asked "Why are you saying 4x4?" he said "you can divide the sixteen towers into groups of four towers each." Suspecting that Gabe was suggesting that the total number of towers was equal to "height times itself," researchers designed a plan to challenge Gabe's thinking in the following session: They would ask students to predict the number of towers three-tall choosing from two colors. If students said, "nine" they would ask them to build the towers. Since students would not find nine towers and could discover that the number of towers had to be even, they hoped that students would abandon the "height x height" inference rule by contradiction. The session unfolded as they predicted.

Figure 1. Gabe (on the left) built four groups of four towers

Analysis. In terms of argumentation, "height x height" can be considered a *warrant* to support the *claim* that there are 16 towers and students needed to understand that it was not valid (object of learning). Working on the three-tall two-colors problem introduced task variation in the form of a counter-example that challenged the "height x height" inference rule opening the possibility of students seeing that it was not valid (lived object of learning), which they did (Lived object of learning). However, just asking students to predict the answer to the three-tall two-color problem did not help them see the contradiction. Asking them to build towers did help them see the contradiction as results did not match prediction. Task variation helped challenge a warrant *empirically*.

DISCUSSION

The results provide insights into how task variation can support argumentation. They show task variation can help build as well as challenge arguments. Teachers are encouraged to provide supports left implicit in students' arguments (Yackel, 2002). The results show that task variation can help students build supports. The results suggest types of variation that support argumentation including sequence of isomorphic tasks, competing solutions, counterexamples, collaboration over tasks, and promoting empirical reasoning. Three types of actions that help teachers support argumentation include direct contributions to arguments, asking questions, and other supportive actions (Conner et al, 2014). Task variation is an example of other actions. The results highlight the importance of using Toulmin's complete scheme to analyze arguments and viewing the purpose of argumentation as reducing and not only eliminating uncertainty (Inglis et al, 2007). The results show that the scheme is useful for analyzing arguments of school students and qualifiers help analyze the development of the arguments.

References

Ayalon, M., & Hershkowitz, R. (2017). Mathematics teachers' attention to potential classroom situations of argumentation. *The Journal of Mathematical Behavior, 49*, 163–173.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education, 41*(4), 351–382.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*, 401–429.

Dewey, J. (1933). How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process, D.C. Heath and Co., Boston

Knipping, C., & Reid, D.A. (2019). Argumentation analysis for early career researchers. In G. Kaiser and N. Presmeg (Eds.), *Compendium for Early Career Researchers in Mathematics Education* (pp. 3-31). ICME-13 Monographs

Kosko, K. W., Rougee, A., & Herbst, P. (2014). What actions do teachers envision when asked to facilitate mathematical argumentation in the classroom? *Mathematics Education Research Journal, 26*, 459–476.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), The emergence of mathematical meaning: Interaction in classroom cultures (pp. 229–269). Hillsdale, NJ: Lawrence Erlbaum.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*, 3–21.

Kilpatrick, J., Swafford, J. and Findell, B. (eds.) (2001). Adding it Up: Helping Children Learn Mathematics, National Academy Press, Washington, DC.

Marton, F. (2015). Necessary Conditions of Learning. New York & London: Routledge. Meyer, J., & Land, R. (2003). Threshold Concepts and troublesome knowledge:

CTM (2000). Principles and standards for school mathematics. Reston, VA: NCTM.

NGA & CCSSO (2010) Common core state standards: Mathematics standards. Washington, DC.

Orgill, M. (2012). Variation Theory. In N. M. Seel (ed.) Encyclopedia of the Sciences of Learning. Springer, Boston, MA. https://doi.org/10.1007/978-1-4419-1428-6_272

Powell, A. B., Francisco, J. M., and Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*, 405–435.

Toulmin, S. E. (2003). The uses of argument (updated ed.). New York: Cambridge University Press. Original work published 1958.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: using variation to structure sense-making. *Mathematical Teaching and Learning, 8*(2), 91–111.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21*, 423–440.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*, 297–321.

SUPPORTING LOWER SECONDARY STUDENTS' FUNCTIONAL THINKING WITH SPECIFIC LEARNING ENVIRONMENTS

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Functional thinking is relevant in mathematics education and everyday life. Despite its relevance, difficulties related to this topic are empirically well documented. Within the project FunThink, learning environments were developed along the four design principles, situatedness, inquiry-based learning, embodiment, and (digital) tools to foster students' functional thinking from primary to upper secondary education. Three learning environments were implemented within the teaching unit of linear functions in two grade eight classes (N=52). Results derived from a pre-post-test study indicate a marginally significant effect on the targeted facets of students' functional thinking concerning the correctness of answers and little changes in students' reasoning.

INTRODUCTION

Functional thinking as thinking in relationships, dependencies, and changes is considered a prototype of a fundamental idea of mathematics education and is relevant from primary to upper secondary education and in everyday life (Vollrath, 1989). Despite its relevance, student difficulties related to this topic are empirically well-documented (Sproesser et al., 2022). So far, various approaches to foster functional thinking have been developed and empirically investigated (e.g., Lichti & Roth, 2018; Stephens et al., 2017) but functional thinking still appears to be challenging for students. Against this background, the Erasmus+ Project FunThink aimed to support students' functional thinking by developing learning environments for different grade levels from primary to upper secondary school along four design principles. These design principles have shown in prior studies, separately or in smaller combinations, to be beneficial for student learning (e.g., Drijvers, 2019; Duijzer et al., 2020) and were now implemented in combination. The present study investigates the effects of learning environments developed for grade 8 on specific facets of student's functional thinking. By facets of functional thinking, we refer to the used operationalization of functional thinking that was developed based on theoretical considerations. The underlying theoretical background and the conceptional background of the implemented learning environments will be described in the following. Subsequently, we present the methods and results of this study and discuss them.

THEORETICAL BACKGROUND

Functional thinking is characterized as a typical way of thinking when dealing with functions (Vollrath, 1989). The close connection of functional thinking and functions is also highlighted by Stephens et al. (2017) who describe functional thinking "as the process of building, describing, and reasoning with and about functions" (p. 144).

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 216–223). PME.

Hence, students are required to develop different perspectives – the so-called aspects of functional thinking – to understand and solve tasks related to this field. These aspects of functional thinking encompass the input-output aspect, the covariation aspect, the correspondence aspect, and the object aspect (Pittalis et al., 2020; cf. Vollrath, 1989). The input-output aspect focuses on a rather operational view of functions, as the manipulation of an input results in a specific output while the underlying relationship is not necessarily focused (Pittalis et al., 2020). The covariational aspect describes the simultaneous variation of two related variables and focuses on a rather dynamic view of functions. In addition to their framework for covariational reasoning, Thompson and Carlson (2017) describe a framework for variational reasoning (i.e., focusing only on one variable while exploring functional relationships). Variational reasoning can be seen as a precursor for covariational reasoning concerning functional thinking and involves recursive strategies, e.g., identifying patterns and extending them (Pittalis et al., 2020). The third aspect, the correspondence one, focuses on the type of relationship between two variables which creates a certain dependency. This aspect is often used for the formal introduction of functions in school (Vollrath, 1989). The object aspect considers functions as mathematical objects that are used for higher-order processes like differentiation or composition (Vollrath, 1989). This indicates that this aspect is often acquired later in the course of schooling whereas the other three aspects are already accessible for students in primary and lower secondary education.

In addition to these four aspects, Pittalis et al. (2020) point out that representing relationships and using these representations in problem-solving situations is important for functional thinking. This is comprehensible, as representations are crucial to access mathematical objects as functions. Such function representations include inter alia graphs, tables, formulas, and situational descriptions. The abstract character of functions, the need of mastering corresponding aspects and representations in combination with an often rather inner mathematical focus in teaching might be reasons for student difficulties in this field (Sproesser et al., 2022) and highlight the importance of comprehensive support of students' functional thinking. In the next section, we sketch how the FunThink project aimed to support functional thinking.

CONCEPTIONAL BACKGROUND OF THE LEARNING ENVIRONMENTS

In the context of the FunThink project, learning environments were developed along four design principles to support students' functional thinking from primary to upper secondary school. These design principles, namely situatedness, inquiry-based learning, embodiment, and (digital) tools, have proven themselves as supportive in student learning and will be described in the following.

The design principle of situatedness implies the use of situations that are meaningful for students as an entry point for learning (Gravemeijer & Terwel, 2000). This framing enables students to build on prior knowledge. The design principle of inquiry-based learning refers to a working and learning habit similar to research (Dorier & Maass, 2020). Students are encouraged to propose hypotheses, find ways to test them, and discuss their results, which support meaningful learning and communication. The design principle of embodiment highlights the close connection between the body and cognition. Hence, learning can be supported if a learner performs movements that are related to a cognitive learning goal, e.g., with a finger on a tablet (Duijzer et al., 2020). The latter example shows that embodiment can be connected to the design principle of (digital) tools. Tools and especially digital ones can extend a person's scope and can therefore support mathematics learning (Drijvers, 2019).

With these design principles in mind and with respect to the curricula, three learning environments were developed for the teaching unit *linear functions*. In this unit, the concept of function itself, as well as linear functions are introduced in grades 7, 8, or 9. These learning environments consider evidence from prior research (e.g., Duijzer et al., 2020; Lichti & Roth, 2018) and will be sketched in the following. *Walking graphs* focuses on distance-time situations and supports covariational reasoning as well as representational changes between situations and graphs. The learning environment *filling vessels* deals with the relationship of volume and height when filling water in vessels. The learning environment *temperature* focuses on the definition of functions including their uniqueness and the direction of the dependency.

RESEARCH QUESTION

The theoretical background and the developed learning environments lead to the following research question: What impact does the intervention (including the three described learning environments) have on specific facets of students' functional thinking? This research question will be investigated with regard to the scored points of students' answers and students' reasoning.

METHOD

Sample

The three above-described learning environments were implemented within the teaching unit *linear functions* in two eighth-grade classes (secondary school, medium track) to investigate the research question. This led to a sample of 52 students (31 female, 21 male; age: M = 13.38 years, SD = 0.53) in the intervention group. The other lessons of this unit were also developed along the described design principles but with less emphasis (not all design principles were included in every lesson). The entire unit was taught by a member of the research team. A third grade eight class at the same school was recruited as control group (N =24; age: M = 13,58 years, SD = 0,65). The control group used their regular school book and was taught by their mathematics teacher. The number of teaching hours and the topics did not differ between the intervention and control groups.

Test Instrument

A test with three items was used in a pre-post design to measure different facets of functional thinking. These items were adapted from earlier studies investigating functional thinking and focused on the above-mentioned aspects of functional thinking. They were not specifically chosen to fit the presented learning environments but rather to evaluate all learning environments within the project. Item 1 refers to a ride of a remote car in which context students are asked to interpret and construct graphs. This item is based on Duijzer et al. (2020) and is displayed in Figure 1. A total of three points were coded if all parts were answered correctly. This item especially focuses on the covariational aspect as students need to describe changes in the graph by coordinating the covariation of distance and time.

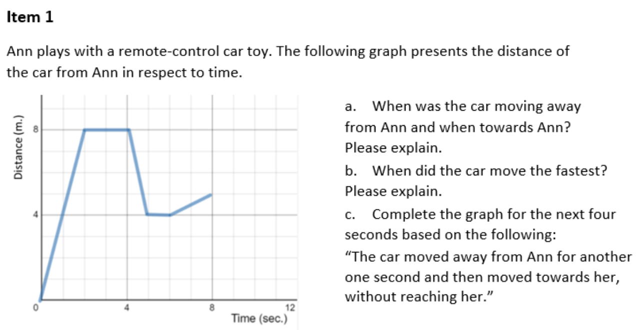

Figure 1: Item 1 of the test instrument (derived from Duijzer et al. (2020, p. 35))

Item 2 was derived from Stephens et al. (2017) and is displayed in Figure 2 (left). Students are asked to find the number of people that can be seated at 8 and 20 tables and the corresponding rule. The questions for concrete pairs of values and the underlying rule suggest referring to the correspondence aspect, yet other strategies are also possible for solving this item. Altogether, one point was scored if both values were indicated correctly. The central element of Item 3 is a function machine that highlights the input-output aspect of functional thinking. This item is based on ideas by Ng (2017) and can be seen in Figure 2 (right). Students are asked to calculate input and output values (one point each) and to find the underlying rule of the function machine. Similar to Item 2, this item can be approached in different aspects.

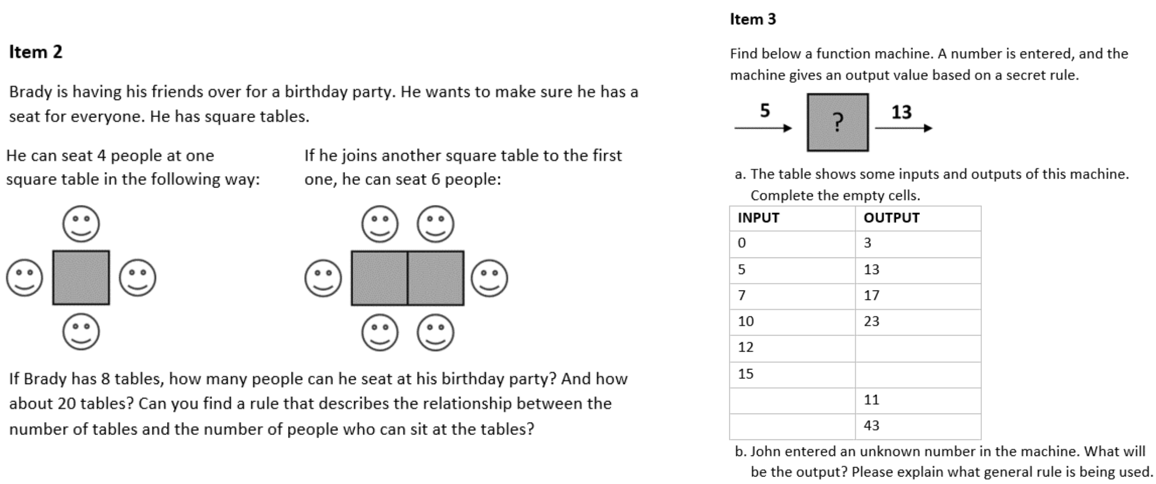

Figure 2: Item 2 (adapted from Stephens et al. (2017, p. 151)) and Item 3 of the test instrument (based on ideas by Ng (2017))

A coding scheme developed within the FunThink project and adapted for this sample was used for analysis. A total of 6 points could be scored according to the descriptions above. In addition, for each item, the used strategies such as recursive or covariational reasoning were coded. The coding was done by two independent coders. Cohen's kappa for interrater reliability was between 0.81 and 1.00. Differences in coding were discussed until consent was reached.

RESULTS AND DISCUSSION

Scored points

In the following, results concerning scored points and reasoning will be presented and discussed. A one-way ANCOVA was conducted to analyze the difference between the intervention and control group on post-test scores controlling for pre-test scores. All prerequisites were met with no significant differences in pre-test scores between the two groups ($t(74) = -0.44$; $p = 0.66$). After adjusting for pre-test scores, post-test scores differ marginally significant between the two groups ($F(1, 73) = 2.87$, $p = 0.095$, partial $\eta^2 = 0.038$). Table 1 shows the means and standard derivations for the scored points for the intervention and control group (unadjusted).

	Intervention group				Control group			
	Pre-test		Post-test		Pre-test		Post-test	
	M	SD	M	SD	M	SD	M	SD
Total	2.23	1.50	2.73	1.52	2.06	1.48	2.10	1.57
Item 1	1.32	0.86	1.68	0.91	1.02	0.89	1.08	0.95
Item 2	0.38	0.43	0.48	0.46	0.48	0.48	0.38	0.47
Item 3	0.52	0.82	0.57	0.83	0.56	0.78	0.65	0.88

Table 1: Statistics of the scored points in pre- and post-test

Table 1 shows a comparably large increase in points for Item 1 for the intervention group with only a small increase for the control group. Therefore, a one-way ANCOVA was conducted for this item separately. Again, all prerequisites were met (t-test: $t(74) = 1.41$; $p = 0.164$). For item 1, after adjusting for pre-test scores, post-test scores differed statistically significant between the intervention and control group ($F(1, 73) = 4.74$, $p = 0.033$, partial $\eta^2 = 0.061$) with a medium effect. The significant difference for Item 1 might be explained as a comparable context was given in the learning environment *walking graphs*. This is in line with prior research (Duijzer et al., 2020) which highlighted the effects of a similar embodied learning environment. In contrast, the textbook used by the control group also thematizes comparable contexts but with less emphasis and without the use of the described design principles (which might explain the small changes for the control group). Tasks similar to the other two items were not specifically focused during the intervention nor in the textbook which might explain the corresponding non-significant effects.

Reasoning

In addition to the scored points, students' reasoning was investigated and corresponding results will be presented in the following. As most changes in terms of scored points could be observed for Item 1 of the test, students' reasoning will be analyzed for this item in detail. For Items 2 and 3, only selected findings will be complemented. Starting with Item 1, the reasoning for parts a and b of this item was coded separately and, in both cases, the number of referred variables (out of the variables "time", "distance", "direction of the line", "speed", "slope") was coded as single variable (use of one of the mentioned variables) or multivariable (use of 2 or 3 of the mentioned variables to justify the answer). The use of one variable can be considered variational reasoning and the use of multiple variables as covariational reasoning respectively (Thompson & Carlson, 2017). The number of answers for each category can be seen in Table 3.

	Intervention group (N=52)		Control group (N=24)	
	Pre-test	Post-test	Pre-test	Post-test
Part a:				
Single variable	28	30	8	13
Multi variable (2)	9	8	7	2
Multi variable (3)	-	1	-	-
Wrong/missing answers	15	13	9	9
Part b:				
Single variable	11	12	3	5
Multi variable (2)	17	23	8	8
Multi variable (3)	-	1	-	-
Wrong/missing answers	24	16	13	11

Table 3: Absolute frequencies of answers for each category in pre- and post-test

The biggest change in reasoning was observed for the intervention group in Part b; the multi variable reasoning (2) increased and the wrong / missing answers decreased from pre- to post-test. In addition to the analysis of frequencies, a Chi-square test was conducted for each part and each group separately but did not show any significant changes in reasoning from pre- to post-test.

Similar to the reasoning in Item 1, the reasoning for Items 2 and 3 did not show any significant changes for both groups. For Items 2 and 3, reasoning was coded along the aspects of functional thinking. Moreover, recursive/variational reasoning was added as an additional category. The use of the object aspect was not expected to be observed in the present grade level as it often develops later in the course of schooling (Pittalis et al., 2020). For the intervention group, the results indicate a slight increase in recursive and correspondence reasoning, as well as a slight decrease in covariational reasoning and incorrect reasoning from pre- to post-test. In the control group, there was a slight increase in recursive reasoning and a decrease in covariational and correspondence reasoning from pre- to post-test. Regarding Item 3, the results show an increase in correspondence reasoning for the intervention group, while for the control group, the reasoning barely changed. A possible explanation for the increase in correspondence reasoning for Items 2 and 3 could be that both items refer to linear functions, which were the focus of the researched teaching unit. The use of the correspondence aspect (e.g., use of function equations) may be an efficient way to solve these items.

CONCLUSION

The presented results indicate a positive effect of the developed learning environments on solving mainly one of the items related to functional thinking. More particularly, this could especially be observed in terms of scored points for Item 1. The changes indicate an improvement in covariational reasoning, an important aspect of functional thinking (Pittalis et al., 2020). In addition, these results confirm previous studies that investigated effects of the design principles in learning environments individually (e.g., Digel & Roth, 2022; Duijzer et al., 2020). The observed development of functional thinking can be seen as positive itself and in addition, it might also be important as a baseline for following topics like quadratic functions. The presented results suggest that a beneficial effect of the learning environments along the design principles can be expected for functional thinking but have also transfer potential for other subject areas. However, the results also show that the change was at most moderate and that significant effects were only recorded if the test items were related to a certain degree to the learning environments. To validate these results and to gain further insights into students' reasoning, the results of further classes, which were also taught by other people, will be evaluated soon. Moreover, the analysis of a more specific test will be complemented and compared to these results, which might give insights into possibly more profound learning gains from the intervention.

Acknowledgment

This study is co-funded by Erasmus+ program of the European Union and the Ludwigsburg University of Education. The European Commission's support for the production of this publication does not constitute an endorsement of the contents, which reflect the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

References

Digel, S., & Roth, J. (2022). Fostering a concept of function with combined experiments in distance and in-class learning. In C. Fernández, S. Llinares, Á. Gutiérrez Rodríguez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 227–234). PME.

Dorier, J.-L., & Maass, K. (2020). Inquiry-based mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., 384-388). Springer.

Drijvers, P. (2019). *Embodied instrumentation: combining different views on using digital technology in mathematics education*. 11th Congress of the European Society for Research in Mathematics, Utrecht, Netherlands. https://hal.archives-ouvertes.fr/hal-02436279

Duijzer, C., van den Heuvel-Panhuizen, M., Veldhuis, M., Boom, J., Doorman, M., & Leseman, P. (2020). Moving towards understanding: Students interpret and construct motion graphs. *Mediterranean Journal for Research in Mathematics Education*, *17*, 25–51.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777–796.

Lichti, M., & Roth, J. (2018). How to foster functional thinking in learning environments using computer-based simulations or real materials. *Journal for STEM Education Research*, *1*(1-2), 148–172. https://doi.org/10.1007/s41979-018-0007-1

Ng, S. F. (2017). Function tasks, input, output, and the predictive rule: How some Singapore primary children construct the rule. In C. Kieran (Ed.), *ICME-13 Monographs. Teaching and learning algebraic thinking with 5- to 12-year-olds: The Global Evolution of an Emerging Field of Research and Practice* (pp. 167–193). Springer. https://doi.org/10.1007/978-3-319-68351-5_7

Pittalis, M., Pitta-Pantazi, D., & Christou, C. (2020). Young students' functional thinking modes: The relation between recursive patterning, covariational thinking, and correspondence relations. *Journal for Research in Mathematics Education*, *51*(5), 631–674. https://doi.org/10.5951/jresematheduc-2020-0164

Sproesser, U., Vogel, M., Dörfler, T., & Eichler, A. (2022). Changing between representations of elementary functions: students' competencies and differences with a specific perspective on school track and gender. *International Journal of STEM Education*, *9*(33). https://doi.org/10.1186/s40594-022-00350-2

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E. & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636

Thompson, P. W., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *The compendium for research in mathematics education* (pp. 421–456). The National Council of Teachers of Mathematics.

Vollrath, H.-J. (1989). Funktionales Denken [Functional thinking]. *Journal Für Mathematik-Didaktik*, *10*(1), 3–37.

STUDENTS' BELIEFS CONCERNING THE NATURE OF MATHEMATICS – ARE THEY DIFFERENT WITH REGARD TO SCHOOL AND UNIVERSITY MATHEMATICS?

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Students' beliefs concerning the nature of mathematics are considered to play a crucial role for a successful transition from school to university mathematics. As school and university mathematics differ considerably, distinguishing between students' beliefs regarding school and university mathematics seems necessary. In this paper, a new questionnaire differentiating beliefs between both facets of mathematics is presented and analysed. Confirmatory factor analysis with data from N=153 students shows that students' beliefs can be distinguished empirically and that students hold significantly different beliefs regarding the nature of school and university mathematics.

INTRODUCTION

High dropout rates from mathematics university programs in the first year reveal that the transition from school to university mathematics is challenging for many students (Dieter & Törner, 2012). Several scholars have argued that students' beliefs concerning the nature of mathematics could have an important impact on students' learning behaviour and thus may be a relevant factor for a successful transition. Moreover, beliefs that are incongruent to the mathematics students face at university may enhance the risk to drop out (Daskalogianni & Simpson, 2001).

However, empirical studies have reported inconsistent results regarding the beliefs that undergraduate students hold at the beginning of their mathematics programs (e.g., Törner & Grigutsch, 1994). In addition, no or only small effects of students' beliefs on students' achievement and dropout behaviour have been reported (e.g., Geisler, 2023). One reason for these inconsistent results could be the differences between school and university mathematics and that – using questionnaires – it is not clear whether students have school or university mathematics in mind, when reporting about their beliefs. In this paper, I present a questionnaire distinguishing between beliefs concerning school mathematics and concerning university mathematics in order to give a differentiated insight which beliefs students hold when entering university mathematics courses.

THEORETICAL BACKGROUND

Beliefs concerning the nature of mathematics

Philipp (2007, p. 259) defines beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true". Beliefs are more cognitive than other affective constructs like emotions and are often described as subjective knowledge (cf. Liebendörfer & Schukajlow, 2017). Beliefs influence

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 224–231). PME.

learning processes in multiple ways as they shape how mathematical tasks are approached and how much effort is put into them. Moreover, they function as filter regulating which information is learned (Philipp, 2007).

Beliefs are always referring to a certain beliefs object which can be mathematics in general but also specific objects like proofs or a topic like geometry (Grigutsch & Törner, 1998). In this paper, I focus on beliefs concerning the nature of mathematics (what one believes how mathematics as a discipline is characterized).

Different systems to classify beliefs regarding the nature of mathematics have been proposed. I follow the approach of Grigutsch and Törner (1998) who describe four main views or aspects of mathematics. The *formalism* aspect highlights the formal rigour and strength – as it appears in formal definitions and proof – as a main characteristic of mathematics. This aspect also stresses the role of abstraction and logical thinking. The *schema* aspect pays attention to schematic and algorithmic facets of mathematics. Persons, holding this belief in an extreme way, see mathematics as a rather static system of (unconnected) rules and formula and as a toolbox containing schematic procedures to solve tasks. In the sense of the *application* aspect, mathematics has practical relevance like applications in other sciences, society and everyday life. The *process* aspect portrays mathematics as a vivid and creative field of research in which it is possible to gain (subjectively) new insights on one's own. Moreover, people with strong process beliefs pay more attention to the process of solving tasks than on the products of these tasks and acknowledge that complex tasks can be solved in multiple ways. According to Grigutsch and Törner (1998) these aspects can be seen as independent dimensions of persons' beliefs and agreeing to all of them to a certain degree is possible. Thus, the aspects have no normative character.

Differences between school and university mathematics

It is a well-known fact that mathematics at school and advanced mathematics at university differ in many facets (for an overview see Engelbrecht (2010) and Gueudet (2008)). New concepts in school are often learnt rather intuitively with many examples and yield on an intuitive understanding while formal definitions are less relevant. Moreover, proofs are only seldom learned at school (in Germany). Typical tasks at school involve rather schematic calculations or the application of concepts and procedures to solve real world problems. Complex problem-solving tasks are used rather seldom (OECD, 2020). In contrast, in university courses new concepts are introduced via formal definitions and a major focus is on rigorous deductive proofs. Moreover, schematic calculations and real-world problems do not play a role in typical tasks. Instead, most tasks are proof related and cannot be solved using schematic procedures (Weber & Lindmeier, 2020).

Following these considerations, beliefs highlighting the *formalism* as well as the *process* aspect seem to fit well to university mathematics and could be less relevant for school mathematics. In contrast, the *application* and *schema* aspect can be considered more appropriate with regard to school mathematics than to university mathematics.

Students' beliefs during the transition from school to university mathematics

Given their influence for learning processes, beliefs that fit to the mathematics students get to know at university are considered beneficial for a successful transition. Sticking to beliefs established at school that are incongruent with the characteristics of university mathematics can be problematic (Daskalogianni & Simpson, 2001).

However, previous studies reported inconsistent results concerning the beliefs first-year students hold (Crawford et al., 1994; Geisler & Rolka, 2018; Törner & Grigutsch, 1994). Moreover, while Crawford et al. (1994) report that students with schema beliefs were less successful in their exams, Geisler (2023) reports only small relations between students' beliefs and their achievement as well as dropout behaviour. One reason for these inconsistent results could be that in these studies, the rather general beliefs object "mathematics" was addressed. Due to the differences between school and university mathematics one can argue that the beliefs object changes during the transition and that school mathematics respectively university mathematics constitute different beliefs objects. Therefore, questionnaires should distinguish between both beliefs objects – otherwise it is not clear which beliefs object students have in mind. Recent studies concerning other affective variables like interest and self-concept show that students clearly distinguish between both objects and that variables focusing university mathematics are better predictors for a successful transition (Rach et al., 2021; Ufer et al., 2017). Thus, distinguishing between school and university mathematics could be a valuable approach regarding beliefs too.

THE CURRENT STUDY

To enable differentiated insights in students' beliefs regarding the nature of school and university mathematics, new questionnaire scales have been developed and analysed.

Operationalization of the new beliefs scales

Most questionnaires that are based on Grigutsch and Törner's (1998) conceptualisation of beliefs regarding the nature of mathematics are shortened versions of their original questionnaire with subscales for the four proposed aspects and the rather general beliefs object "mathematics" (with items like "Mathematics helps to solve daily tasks and problems.", e.g., Liebendörfer & Schukajlow, 2017). To operationalize differentiated scales for the beliefs objects school and university mathematics, the items from the shortened scales of Liebendörfer and Schukajlow (2017) have been adapted and rephrased, resulting in items that explicitly refer to "school mathematics" respectively "university mathematics". Thus, every item exits in two versions. Table 3 on page 7 gives an overview of the developed scales and example items.

Research questions

The purpose of the study at hand is twofold: On a rather methodological level the dimensionality of the developed scales and thus the structure of students' beliefs seems particular relevant. In order to get a more differentiated insight in students' beliefs, possible differences in the beliefs students hold regarding school and university mathematics can be identified. This leads to the following research questions:

1. Which structure do students' beliefs concerning the nature of school and university mathematics have? Is it possible to differentiate the *formalism*, *schema*, *application* and *process* aspect empirically regarding the beliefs objects school and university mathematics?

Given that the proposed aspects are (theoretically) differently relevant for school and university mathematics (e.g., process and formalism aspect more related to university mathematics and application as well as schema aspect more relevant with regard to school mathematics), I expect that students will hold different beliefs regarding both beliefs objects. Thus, their beliefs will be structured in eight dimensions and a model differentiating all aspects between school and university mathematics will best describe the data (H1).

2. Which beliefs regarding the nature of school and university mathematics do first year students hold and which differences between both sets of beliefs can be found?

Based on the aforementioned characters of school and university mathematics, I expect that students will hold stronger schema and application beliefs regarding school mathematics than regarding university mathematics (H2). Contrary, the formalism aspect as well as the process aspect will be stronger regarding university mathematics than regarding school mathematics (H3).

Methods

153 first-year students (*M*(age)=20; 58 % male) attending real analysis lectures at two public German universities filled out the new questionnaire during the second week of the winter term. These students were enrolled in pure mathematics bachelor programs or a bachelor program for upper secondary pre-service teachers. All items have been answered on a 5-point likert scale ranging from *1 = not true at all* to *5 = totally true*.

Item analysis and confirmatory factor analysis (CFA) have been used to inspect the developed scales and to answer research question 1. The item analysis let to the deletion of two items from both intended *schema* scales due to low reliability as well as a low item-total correlation in the possible models. Moreover, both items caused convergence problems in the CFA models.

Four possible models (see figure 1 for an overview, possible correlations among factors are not visualized to enhance readability) have been tested with CFA in Mplus 8. As not all items were normally distributed, robust maximum likelihood estimation was used. Model 1 contains eight independent factors: *formalism*, *schema*, *application* and *process* aspect each differentiated regarding the two beliefs objects school and university mathematics – as proposed in H1. Model 2 pays attention to the possibility that the four aspects proposed by Grigutsch and Törner (1998) do not differentiate in different factors for school and university mathematics. Model 3 tests the possibility that beliefs can be differentiated in those regarding the beliefs objects school and university mathematics but without paying attention to the four proposed aspects. In model 4, a single factor containing all items regardless the aspects and the beliefs object (school vs. university mathematics) was modelled. To answer research question 2, eight scales (as intended in model 1) have been formed and the means have been compared using paired *t*-tests.

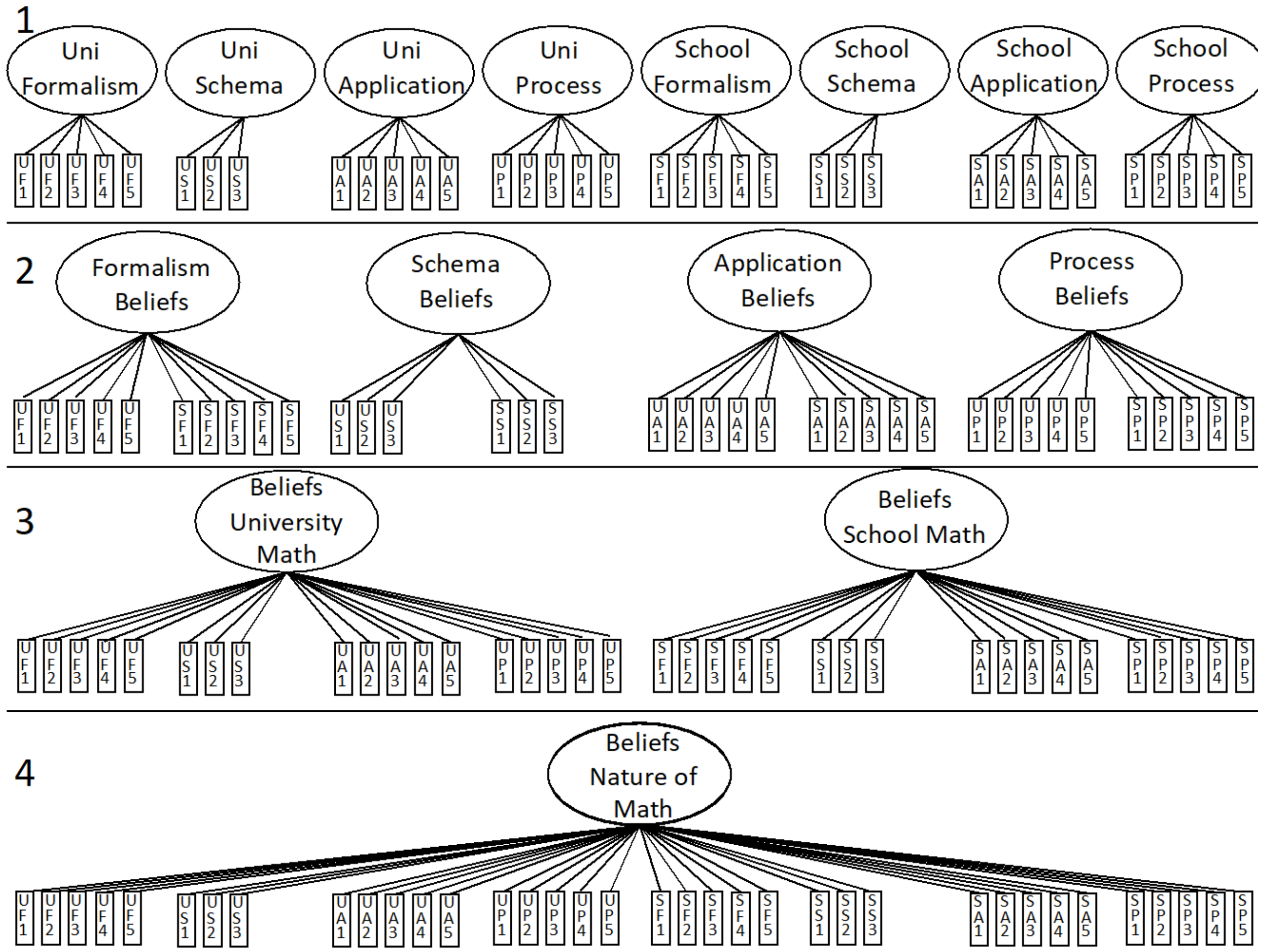

Figure 1: Overview of the tested models for the structure of the beliefs questionnaire

RESULTS

Model	χ^2	*p*	χ^2/df	RMSEA	CFI	SRMR	AIC	BIC
1	837	<.001	1.48	0.056	0.81	0.083	14486	14468
2	1520	<.001	2.59	0.102	0.36	0.155	15137	15121
3	1291	<.001	2.18	0.088	0.52	0.118	14919	14905
4	1615	<.001	2.72	0.106	0.30	0.137	15233	15219

Table 1: Overview of the model fit for the tested models (BIC sample-size adjusted)

The CFAs reveal that the intended model with eight factors (model 1) differentiating each aspect (*formalism, schema, application, process*) for both beliefs objects (school and university mathematics) best fits the data, having the lowest AIC and BIC values (see table 1). Moreover, model 1 has better descriptive fit values than the concurrent models, confirming H1. However, the fit of model 1 is not perfect. Whereas RMSEA and SRMR indicate a good or acceptable fit, the CFI value is rather low (Schermelleh-Engel et al., 2003). The chi-square test is still significant which could be due to the small sample size. In this case, Schermelleh-Engel et al. (2003) suggest to interpret the proportion of the chi-square value and the degrees of freedom. $\chi^2/df < 2$ indicates a good fit for model 1. All factor loadings are significant with $p<.001$.

The intended eight-dimensional model seems to be most appropriate to describe the structure of students' beliefs and shows that students at the beginning of their studies already differentiate between school and university mathematics, when rating their beliefs. Therefore, eight sub-scales have been formed with four scales referring to the beliefs object university mathematics and four scales referring to school mathematics. Both subscales for the *schema* aspect have rather low reliability while all other scales have satisfying reliability (see table 2 for reliability and descriptive statistics).

Beliefs Object:	**University Math**			**School Math**			
	M	*SD*	α	*M*	*SD*	α	Cohen's *d*
Formalism	3.97	0.69	.78	3.16	0.72	.69	0.89***
Schema	3.12	0.85	.55	4.12	0.62	.58	0.91***
Application	2.78	0.88	.79	3.50	0.82	.80	0.62***
Process	3.81	0.70	.76	3.57	0.76	.76	0.23**

Table 2: Descriptive statistics, reliability (cronbach's α) and results of the paired *t*-tests, ***p*<.01, ****p*<.001, answers between *1 = not true at all* and *5 = totally true*

With regard to research question 2, the paired *t*-tests revealed clear differences in students' beliefs concerning the nature of school mathematics and the nature of university mathematics. Confirming H2, students agreed significantly stronger to the *schema* and *application* aspect regarding school mathematics than regarding university mathematics with middle to strong effect size (table 2). As expected in H3, students agree more to the *formalism* and *process* aspect with regard to university mathematics than regarding school mathematics. The differences in the *formalism* aspect reveal a strong effect, whereas the effect size for the *process* aspect is small (table 2).

DISCUSSION

The results of the confirmatory factor analysis show that it is possible to empirically differentiate students' beliefs regarding school and university mathematics in all four proposed aspects. This result is similar to previous results of studies differentiating other affective variables like interest and self-concept regarding school and university mathematics (Rach et al., 2021; Ufer et al., 2017). Thus, students seem to hold different beliefs concerning the two beliefs objects school and university mathematics. This is also confirmed by the strong differences especially for the *formalism* and *schema* aspect. These differences in students' beliefs fit to the theoretical considerations that formalism plays no central role in school mathematics but is mandatory for university mathematics and reflect, that students clearly notice these differences. Moreover, students seem to see school mathematics stronger related with schematic calculations than university mathematics. This is in line with empirical results that schematic tasks are frequently used in school (OECD, 2020) but loose importance in university mathematics courses (Weber & Lindmeier, 2020).

Limitations of the study lie in the rather small sample as well as the used items. The model fit is not perfect which could be due to the rather small sample. Specially the items for the *schema* aspect should be rephrased because some had to be excluded and the resulting scales have rather low reliability. Thus, the results concerning the *schema* aspect should be interpreted cautious and replicated within a larger sample.

Not differentiating between school and university mathematics in former questionnaires could be a reason for inconsistent results regarding the beliefs that first-year students hold. The new developed questionnaire enables more nuanced insights in students' beliefs. Moreover, differentiating between beliefs regarding school and university mathematics could be valuable for predicting success in mathematics programs in future studies - as differentiating other affective variables was in previous studies (e.g., Rach et al., 2021).

Beliefs Object:	**University Math**		**School Math**	
	#	Example Item	#	Example Item
Formalism	5	Of major importance for math, as it is done at university, is its logic rigour and precision.	5	Mathematical thinking in school is characterized by abstraction and logic.
Schema	5/3	Nearly all math problems in university can be solved by directly using known rules, formula and routines.	5/3	Math, as it is done in school, contains learning, remembering and application.
Application	5	Many aspects of university math have a practical benefit or direct applications.	5	In school math one works on tasks that have a practical use.
Process	5	Mathematical tasks and problems at university can be solved correctly in different ways.	5	Doing school math means understanding facts, seeing relations and having ideas.

Table 3: Overview of the developed scales with number of items (*Schema*: 5 items developed, 2 items excluded due to low reliability and low item-total correlation)

References

Crawford, K., Gordon, S., Nicholas, J., & Prosser, M. (1994). Conceptions of mathematics and how it is learned: The perspectives of students entering university. *Learning and Instruction, 4*(4), 331–345.

Daskalogianni, K., & Simpson, A. (2001). Beliefs overhang: The transition from school to university. In J. Winter (Ed.), *Proc. of the British Congress of Mathematics Education* (vol. 21, pp. 97–108).

Dieter, M., & Törner, G. (2012). Vier von fünf geben auf. [Four out of five are giving up.] *Forschung & Lehre*, *19*(10), 826–827.

Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *IJMEST*, *41*(2), 143–154.

Geisler, S. (2023). What role do students' beliefs play in a successful transition from school to university mathematics? *IJMEST, 54*(8), 1458–1473.

Geisler, S., & Rolka, K. (2018). Affective variables in the transition from school to university mathematics. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics* (pp. 507–516). University of Agder and INDRUM.

Grigutsch, S., & Törner, G. (1998). *World views of mathematics held by university teachers of mathematics science (Preprint Series No. 420)*. University of Duisburg.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *ESM*, *67*(3), 237–254.

Liebendörfer, M., & Schukajlow, S. (2017). Interest development during the first year at university: do mathematical beliefs predict interest in mathematics? *ZDM*, *49*(3), 355–366.

OECD. (2020). *Global teaching InSights: A video study of teaching*. OECD Publishers.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematcs teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 257–315). Information Age Publication.

Rach, S., Ufer, S., & Kosiol, T. (2021). Die Rolle des Selbstkonzepts im Mathematikstudium – Wie fit fühlen sich Studierende in Mathematik? [The role of self-concept in mathematics university programs.] *Zeitschrift für Erziehungswissenschaft*, *24*(6), 1549–1571.

Schermelleh-Engel, K., Moosbrugger, H., & Müller, H. (2003). Evaluating the fit of structural equation models: Tests of significance and descriptive goodness-of-fit measures. *Methods of Psychological Research Online*, *8*(2), 23–74.

Ufer, S., Rach, S., & Kosiol, T. (2017). Interest in mathematics = interest in mathematics? What general measures of interest reflect when the object of interest changes. *ZDM*, *49*(3), 397–409.

Weber, B.-J., & Lindmeier, A. (2020). Viel Beweisen, kaum Rechnen? Gestaltungsmerkmale mathematischer Übungsaufgaben im Studium. [Many proofs, less calculating? Features of mathematics tasks at university]. *Mathematische Semesterberichte*, *39*(2), 223–245.

Törner, G., & Grigutsch, S. (1994). „Mathematische Weltbilder" bei Studienanfängern - eine Erhebung. [Mathematical worldviews of first-year students.] *JMD*, *15*(3–4), 211–251.

MATH FOR TEACHING OR UNIVERSITY? - PRESERVICE TEACHERS' MOTIVATION IN THEIR FIRST STUDY YEAR

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Many preservice teachers lose their motivation in mathematics during their first year of study, displaying an unfavorable view of not being interested in mathematics. Given the evidence that they are not only interested in mathematics, but teaching as well, we operationalized career- and subject-specific dimensions in their motivation for mathematics, using expectancy-value-cost theory. Findings based on 209 higher-secondary and primary preservice teachers show a great fit between the theoretically anticipated model and the empirical data. The motivational development based on those dimensions shows a decrease for subject-specific interest but an increase for subject-specific relevance, indicating a shift from intrinsic to extrinsic motivation, while career-specific values remain stable in the first year. Practical implications how to address career-specific values in mathematics teacher education are being discussed.

PRESERVICE TEACHERS MOTIVATION IN MATHEMATICS

Motivation is a crucial factor within teacher education. Among other things, motivation is a predictor of study retention (Schnettler et al., 2020), study satisfaction (Rach, 2014), and later learning success (Biermann et al., 2019), as well as the quality of preservice teacher's (PST) future teaching (Biermann et al., 2019). In mathematics, however, many PST seem to quickly lose their motivation during the first year (Rach, 2014). While such cooling-out effects have been discussed for all mathematics students who are facing cognitive challenges in the transition to mathematics at university (Rach, 2014), PST take up a specific role: Higher-secondary PST (teaching grades 5-13, students ages 10-19) for example more strongly lose their interest during the transition, compared to major students (Rach, 2014). They report being less satisfied with their overall studies and claim most of the mathematical contents as irrelevant for them (Gildehaus & Liebendörfer, 2022). They report copying homework more often and using more surface learning strategies (Gildehaus & Liebendörfer, 2022). Similar findings are evident for primary PST (teaching grades 1-4, students ages 6-10). They report a high level of dissatisfaction and a lack of perceived relevance of the content as well (Coppola et al., 2012).

Current approaches explain those findings by further differentiating PST interests and motivation e. g., by differentiating interest in the subject they study from pedagogical interest in teaching in general. In such operationalizations, primary PST for example report higher interest for pedagogy, and teaching in general than for mathematics, while higher-secondary PST report equal interest for both dimensions (Fray & Gore,

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 232–239). PME.

2017; Gildehaus & Göller, 2023). Such general differentiations of career- and subject-specific dimensions seem promising to better describe and understand PST motivation in mathematics and subsequent variables, e.g., lower interest in mathematics is related to lower study satisfaction (Kosiol et al., 2019).

Current differentiations of students' motivation, however, do not take up a mathematics-specific perspective. This may be of relevance as the object of mathematics changes for many students during the transition to university. In the subject-specific dimension, interest may thus most naturally change if the object of interest is changed (Ufer et al., 2017). Qualitative findings suggest that also the career-specific dimension may relate specifically to the subject: PST of different subjects may value different actions as relevant for them and their teaching, e. g., music PST report high value for orchestra management, while mathematics PST report explaining mathematics to someone as important and motivating for them (Gildehaus et al., 2023). We thus assume that differentiating mathematics-specific dimensions of motivation could help to better understand students' motivation and its connection to their learning and participation. In the following, we aim to operationalize PST motivation by differentiating career- and subject-specific dimensions specifically for mathematics. We further aim to analyze meaningful variations in PST motivation across these dimensions: their development throughout these different dimensions, as well as possibly differences between higher-secondary and primary PST.

THEORETICAL BACKGROUND

We frame our operationalization within Expectancy-Value-Cost theory, which has been well approached to frame multidimensional motivational aspects into one coherent model (cf. Fray & Gore, 2018).

Expectancy-Value-Cost Theory (EVC)

The EVC assumes that a person's motivation is directly related to three beliefs (Barron & Hulleman, 2015): the person's expectation of success (Can I do this?), the importance or value that the person attaches to different options (Do I want to do this?), and the perceived disadvantages ("Costs") of the option (What is stopping me?). Since we mainly aim to investigate PST motivation in terms of what they value around mathematics, we are focusing on values in the following: Individuals may value an option because it is known to be fun (*intrinsic value*; e.g., studying mathematics in a teaching degree program is fun). Likewise, an option can be perceived as significant for one's identity (*attainment value*; e.g., if one sees oneself as a mathematics teacher and therefore values the mathematics teaching degree program). Furthermore, an option can be perceived as useful for current or future goals (*utility value*; e.g., studying to become a mathematics teacher as useful for the future). While intrinsic and attainment value are usually associated with intrinsic motivation, utility value is related to extrinsic motivation (Barron & Hulleman, 2015).

We further assume that motivation is directed towards various objects (Schukajlow et al., 2023). These objects can be of different types and located at different hierarchical levels. Thus, PST can report motivation for mathematics in general but also for objects around mathematics teaching, such as explaining mathematics to someone (Gildehaus et al., 2023). In the following, we outline the state of the art on current differentiations of objects in PST motivation in mathematics, from an EVC perspective.

Motivation for becoming a teacher and studying mathematics at university

For *intrinsic value,* Kunter et al. (2013) differentiated enthusiasm for mathematics and mathematics teaching, indicating that a general differentiation of mathematics and mathematics teaching can be operationalized. However, they did not operationalize different objects of mathematics. This was done by Ufer et al. (2017), who empirically distinguished interest in mathematics as known in school and interest in mathematics as known in university. The same authors further analyzed mathematics students' interest development throughout the first year: While interest in school mathematics mainly remained stable, the interest in university mathematics significantly dropped, specifically for PST (Kosiol et al., 2019). This distinction between school and university mathematics also seems revealing in qualitative research, where PST themselves distinguish between enjoyment of school- and university related mathematics (Gildehaus et al., 2023).

For *attainment value*, no operationalized differentiations around mathematics exist to our knowledge. Thus, we conducted a preliminary study indicating that PST identify themselves with different practices than major students and thus value them differently (Gildehaus & Liebendörfer, 2022). In a subsequent analysis, a differentiation between well explaining and delivering mathematics to someone (as important for one's identity as a future mathematics teacher) and deeply understanding complex mathematics (as important for one's identity as a current mathematics student), was identified (Gildehaus et al. 2023). Piloting studies showed that these two dimensions may also be operationalized (Gildehaus & Göller, 2023).

Utility value for mathematics has been well discussed in terms of the perceived relevance of mathematical content for future teachers. Eichler and Isaev (2023), for example, operationalized mathematics teaching-specific utility value and analyzed how this strongly decreased during the first study year. In addition, Hernandez-Martinez and Vos (2018) described that mathematical content can also be considered highly relevant for the exams and the formal degree while being perceived as not relevant for the future career. Guse et al. (2023) operationalized such a distinction, differentiating between the perceived short-term (for exams) and long-term (for the future career) relevance of content. Such distinctions can also be found in qualitative studies, being brought up by the PST themselves (Gildehaus et al., 2023).

Given these findings around different objects in PST motivation for mathematics, we aim to frame and operationalize PST motivation within a career- and subject-specific dimension the following and thus pose two research questions:

Research Questions

RQ1: Can career- and subject-specific dimensions in PST values for mathematics be operationalized, empirically differentiated, and reliably measured?

RQ2: How do the differentiated career- and subject-specific values develop throughout the first study year and how do higher-secondary and primary PST differ in them?

METHOD

Our sample was recruited at a medium-sized public university in Germany. The PST were enrolled in mathematics teacher education programs for either higher-secondary school or primary school. A total of $n = 209$ PST (73.7% female, 30.6% higher-secondary PST, mean age 19.9) participated in the first questionnaire in the middle of the first semester (T1). In total, $n = 81$ of these PST could be matched with the second questionnaire they filled out in the middle of the second semester (T2; 71.6% female, 22.2% higher-secondary PST).

The PST were surveyed during their main lectures using an online questionnaire. Based on the current findings stated earlier, we anticipated a career- (C) and subject-specific (S) dimension for each value (intrinsic, attainment, utility). Three to four items for each dimension were developed. *Intrinsic value* was differentiated towards mathematics as known from school (IV-C) and mathematics as known in university (IV-S; based on Ufer et al. 2017). *Attainment value* was operationalized in terms of different values and practices relating to different identities: The value of explaining mathematics to someone (AV-C) and the value of deeply understanding mathematics (AV-S). *Utility value* was not differentiated in terms of the object but in terms of the relevance anticipated to the object of the mathematical contents at the university. This could be perceived as relevant either for the future career as a teacher (UV-C) or the upcoming exams at university (UV-S). All items were answered on a six-point Likert scale ("1 - strongly disagree", "6 - strongly agree"). Both questionnaires were identical at both time points. Examples can be found in Table 1.

Value	Example	Items	α T1	α T2
Intrinsic Value Career (IV-C)	I enjoy the mathematics in school.	4	.95	.96
Intrinsic Value Subject (IV-S)	I enjoy the mathematics at university.	4	.89	.85
Attainment Value Career (AV-C)	I want to be someone who is very good at explaining mathematics.	4	.92	.89
Attainment Value Subject (AV-S)	I want to be someone who deeply understands mathematics.	4	.90	.87
Utility Value Career (UV-C)	The mathematical content of my studies is useful for my later career.	3	.89	.90

Utility Value Subject (UV-S)	The mathematical content of my studies is only useful to pass the exams.	3	.71	.68

Table 1: Operationalized Values, Examples, and Cronbach's α

The data analysis for RQ1 focused on confirming the assumed structure of two dimensions for each of the three values based on a confirmatory factor analysis of the data of T1. As it is not possible for us to examine all theoretically conceivable models in confirmatory factor analyses, an exploratory factor analysis was also carried out in order to identify any models that had not yet been theoretically considered (Kaplan, 2004). For the scale evaluation, discriminatory power (corrected item-scale correlation) and internal consistency (Cronbach's α) were used. For RQ2, repeated measures ANOVAs were conducted to investigate differences between T1 and T2 as well as between the two study programs.

RESULTS

For RQ1, the confirmatory factor analysis provided acceptable to good model fit values of the theoretically assumed 6-factor model ($\chi^2(194) = 353$, $p <. 001$, CFI = .96, TLI = .95, RMSEA = .06, SRMR = .05). With regard to explorative factor analysis, the Kaiser-Meyer-Olkin measure with 0.87 showed good suitability of the sample (Kaplan, 2004). The parallel analysis identified six factors corresponding to the theoretically assumed scales. All items loaded with at least .57 on the respective factors. Cross-loadings greater than .30 were not present. The scale definition along the identified six dimensions also provided good internal consistency for T1 as well as T2 (for Cronbach's α, see Table 1). The discriminatory power was also higher than .50 for all items and thus spoke in favor of six one-dimensional scales (Kaplan, 2004).

	All students (n=81)				HSecond. PST (n=18)				Primary PST (n=63)			
	T1		T2		T1		T2		T1		T2	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
IV-C	4.32	*1.41*	4.31	*1.35*	5.65	*0.54*	5.65	*0.44*	3.95	*1.26*	3.93	*1.27*
IV-S	3.31	*0.98*	2.90	*1.00*	3.66	*0.89*	3.44	*0.75*	3.21	*0.98*	2.75	*0.98*
AV-C	5.44	*0.71*	5.30	*0.74*	5.69	*0.57*	5.63	*0.48*	5.38	*0.74*	5.21	*0.79*
AV-S	3.84	*1.01*	3.64	*1.01*	4.19	*0.86*	3.74	*0.97*	3.74	*1.01*	3.61	*1.02*
UV-C	3.47	*1.31*	3.46	*1.29*	3.96	*1.54*	3.41	*1.25*	3.35	*1.18*	3.47	*1.26*
UV-S	3.80	*1.20*	4.46	*1.15*	3.02	*1.30*	4.42	*1.08*	4.05	*1.01*	4.47	*1.16*

Table 2: Means and Standard Deviation for Values at T1 and T2 differentiated by Study Programs

Regarding RQ2, Table 2 displays the means and standard deviation for the Values at T1 and T2. The differences in the means of the Values between T2 and T1 are illustrated in Figure 1. Numbers below zero indicate a decrease in this Value from T1 to T2, and numbers above zero indicate an increase of the respective Value from T1 to T2. The ANOVA showed no significant effect of Time for Intrinsic Value Career ($p = .74$), but a significant effect of the Study Program ($F(1,73) = 28.6$, $p < .001$): Higher-Secondary PST report higher Intrinsic Value Career with big effect size (η_p^2=.28). For Intrinsic Value Subject, both, effects of Time ($F(1,71) = 11.8$, $p < .001$) and Study Program ($F(1,71) = 5.1$, $p = .027$) are significant: Intrinsic Value Subject decreases with medium effect size from T1 to T2 ($\eta_p^2 = .14$) and Higher-Secondary PST report higher Intrinsic Value Subject than Primary PST, but with low to medium effect size ($\eta_p^2 = .07$). For Attainment Value Career, only the Study Program shows a significant effect ($F(1,74) = 4.2$, $p = .045$), but not the Time ($p = .196$): Higher-Secondary PST report higher Attainment Value Career ($\eta^2 = .05$). For Attainment Value Subject, as well as for Utility Value Career, we could not find any significant effects ($p > .126$ in all cases). The ANOVA for Utility Value Subject showed a significant effect of Time ($F(1,63) = 6.5$, $p = .013$), indicating that this Value increased from T1 to T2 with low effect size ($\eta^2 = .10$). There is no effect of the Study Program ($p = .082$), but a significant interaction effect of Time and Study program ($F(1,63) = 4.08$, $p = .048$), indicating that this Value increases for Higher-Secondary PST more over time than for Primary PST ($\eta^2 = .06$).

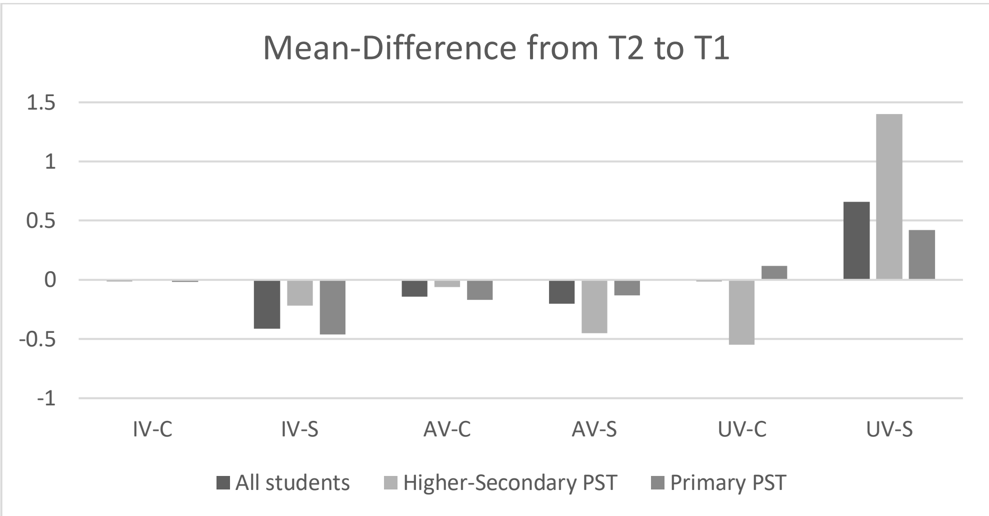

Figure 1: Mean-difference from T2 to T1: Below zero indicates decrease from T1 to T2 and above zero indicates increase from T1 to T2

DISCUSSION

PST seem to often lose their motivation for mathematics during their first year, displaying an unfavourable view of not being interested in mathematics and participating less engaged (Guse et al., 2023). Following the evidence that they are not only interested in mathematics but mathematics teaching as well, we aimed to operationalize career- and subject-specific dimensions in their values for mathematics (RQ1). We further analyzed their motivational development and differences between higher-secondary and primary PST (RQ2) to see if our instrument may uncover meaningful variations.

Findings for RQ1 showed that such a mathematics-specific differentiation of career- and subject-specific dimensions in PST values could be operationalized and empirically confirmed for our sample in all values (intrinsic, attainment, utility). This corresponds to existing differentiations of career- and subject-specific interest (Fray & Gore, 2018; Guse et al., 2023) but shows that the operationalization of mathematics-specific values is possible as well as going beyond interest and intrinsic value.

Findings for RQ2, showed a decrease in subject-specific intrinsic value but not in the career dimension, what is in line with current research (Kosiol et al., 2019). In contrast, we did not find significant decreases in attainment values. Specifically, for subject-specific attainment value this indicates that PST still value deeply understanding mathematics, even when enjoying it less. Contrary to Eichler and Isaev (2023), we found no significant decrease in career-specific utility value, but a descriptive one for the higher-secondary PST. This would be in line with Eichler and Isaev (2023), who reported decreases in relevance mainly for higher-secondary PST. We further observed an increase in subject-specific utility value, indicating that PST may, while not being interested in the contents at university, still perceive them as highly relevant to pass their exams. While this suggests a (probably problematic) shift from intrinsic to extrinsic motivation, it does, in contrast to current findings, not indicate a general loss of motivation. Given the differences between study programs, higher-secondary PST reported higher intrinsic value for career and subject, which is in line with current findings (Gildehaus & Göller, 2023). However, higher-secondary PST also report higher attainment value career, what highlights the importance of the mathematics-specific operationalization used in this study, since, in general, pedagogical interest was found to be higher for primary PST in other studies (Fray & Gore, 2017). We may note though, that our study shows clear limitations given the overall small sample at T2 and specifically the very small subgroup of higher-secondary PST, which restricts the meaning of our findings for RQ2. From a descriptive point of view, it seems that specifically higher-secondary PST do actually also lose their motivation in attainment value subject, as well as utility value career.

Our findings underline the current discussion around motivation in mathematics education: it is the object of motivation that matters (Schulkajlow et al., 2023). The observed shift in PST motivation for the subject may possibly explain their somewhat disengaged participation, what future studies may analyze concretely. Yet, it seems promising, that PST overall intrinsic motivation for career remains rather stable. Practical implications could for example address the attainment value career, by using the valued action of explaining things to each other in university mathematics teaching. Current approaches showing intersections between school and university contents also seem promising, given our findings.

References

Barron, K. E., & Hulleman, C. S. (2015). Expectancy-Value-Cost Model of Motivation. In J. D. Wright (Ed.), *International. encyclopedia. of the social & behavioral sciences* (pp. 503-509). Elsevier.

Biermann, A., Dörrenbächer-Ulrich, L., Grassmé, I., Perels, F., Gläser-Zikuda, M., & Brünken, R. (2019). Hoch motiviert, engagiert und kompetent: Eine profilanalytische Untersuchung zur Studien- und Berufswahlmotivation von Lehramtsstudierenden. *Zeitschrift Für Pädagogische Psychologie*, *33*(3-4), 177-189.

Coppola, C., Di Martino, P., Pacelli, T. & Sabena, C. (2012). Primary teachers' affect: a crucial variable in the teaching of mathematics. *Nordic Stud. in Math. Ed. 17* (3), 107–123.

Eichler, A., & Isaev, V. (2023). Improving Prospective Teachers' Beliefs About a Double Discontinuity Between School and University Mathematics. *JMD*, *44*(1), 117-142.

Fray, L., & Gore, J. (2018). Why people choose teaching: A scoping review of empirical studies, 2007–2016. *Teaching and Teacher Education, 75*, 153–163.

Gildehaus, L., & Liebendörfer, M. (2022). How to educate preservice mathematics teachers? - Identity perspectives in a mixed concurrent setting. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proc. 12th Conf. of European Research in Mathematics Education* (pp. 1739-1747). CERME.

Gildehaus, L., & Göller, R. (2023). Ein Fragebogen zur Erfassung fach- und berufsbezogener Werte für die Studienwahl von Mathematikstudierenden. In I.-P. Goethe-Universität Frankfur (Ed.), *BzMU 2022* (pp. 761-764). WTM.

Gildehaus, L., Liebendörfer, M., & Schukajlow, S. (2023). Preservice teachers' mathematics-related values and expectancy of success in their first study year. *Quadrante, 32*(2), 25-48.

Guse, S., Kienhues, D., & Jucks, R. (2023). Math is cool - for school? Insights into preservice teachers' views on the mathematics discipline between an academic and a job-related focus. *Teaching and Teacher Education*, *132*, 104247.

Hernandez-Martinez, P., & Vos, P. (2018). "Why do I have to learn this?" A case study on students' experiences of the relevance of mathem. modelling. *ZDM 50*(1-2), 245-257.

Kaplan, D. (2004). *The handbook of quantitative methodology for the social sciences*. Sage.

Kosiol, T., Rach, S., & Ufer, S. (2019). (Which) Mathematics Interest is Important for a Successful Transition to a University Study Program? *International Journal of Science and Mathematics Education*, *17*(7), 1359-1380.

Kunter, M. (2013). Motivation as an Aspect of Professional Competence: Research Findings on Teacher Enthusiasm. In M. Kunter, J. Baumert, W. Blum, …, & M. Neubrand (Eds.), *Cognitive Activation in the Mathematics Classroom* (pp. 273-289). Springer US.

Schnettler, T., Bobe, J., Scheunemann, A., Fries, S., & Grunschel, C. (2020). Is it still worth it? Applying expectancy-value theory to investigate the intraindividual motivational process of drop out from university. *Motivation and Emotion 44(4), 491–507.*

Schukajlow, S., Rakoczy, K., & Pekrun, R. (2023). Emotions and motivation in mathematics education: Where we are today and where we need to go. *ZDM 55*(2), 249-267.

Rach, S. (2014). *Charakteristika von Lehr-Lern-Prozessen im Mathematikstudium: Bedingungsfaktoren für den Studienerfolg im ersten Semester.* Waxmann.

Ufer, S., Rach, S., & Kosiol, T. (2017). Interest in mathematics = interest in mathematics? What general measures of interest reflect when the object changes. *ZDM*, *49*(3), 397-409.

STUDY SATISFACTION, PROGRAM CHANGE, AND DROPOUT INTENTION OF MATHEMATICS PRESERVICE TEACHERS FROM AN EXPECTANCY-VALUE THEORY PERSPECTIVE

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In this paper, we aim to better understand the relations of mathematics preservice teachers' mathematics-specific expectancy, values, and costs with their intention to drop out or change their study program as well as with their study satisfaction. Based on data from 209 mathematics preservice teachers, we analyze a structural equation model that highlights the importance of students' expectancy for success as well as the mediating role of students' emotional cost for dropout intention, study program change intention, and study satisfaction. These findings have theoretical and practical implications, which are discussed.

INTRODUCTION

Mathematics teachers are currently in high demand in Germany (Klemm, 2020). At the same time, study dropout is comparatively high in mathematics-related study programs, often already in the first year of study (Geisler, Rach, et al., 2023; Neugebauer et al., 2019). This means that understanding students' decision to drop out of mathematics preservice teacher study programs is of theoretical and practical interest to society. Besides performance difficulties, a lack of motivation is one of the main reasons being interdisciplinary discussed for study dropout (Chen, 2013; Geisler, Rach, et al., 2023; Neugebauer et al., 2019). One common conceptualization of motivation is that of expectancy and values. Accordingly, study dropout intention has recently also been studied from an expectancy-value theory perspective (Eccles & Wigfield, 2020), whereby values, especially intrinsic and attainment value, as well as costs, have been shown to predict study satisfaction and dropout intention (Messerer et al., 2022; Schnettler et al., 2020).

While most of these studies focused on mathematics students and their motivation in general, less is known about mathematics preservice teachers' specific motivation and its relation to study satisfaction and dropout intention. Recent studies indicate that cost, e.g., in terms of frustration (Göller & Gildehaus, 2021), as well as utility, e.g., in terms of perceived relevance of the mathematical contents for their teaching profession (Eichler & Isaev, 2022; Gildehaus & Liebendörfer, 2021), may play a significant role. Furthermore, qualitative studies suggest that preservice teachers may value actions around mathematics that are directly related to the teaching profession, such as explaining mathematics to someone (Gildehaus et al., 2023). If we aim to understand preservice teachers' dropout intentions or study satisfaction, it seems relevant to consider these specific dimensions of their motivation. In the following, we thus

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 240–247). PME.

investigate mathematics preservice teachers' dropout and program change intention as well as their study satisfaction from an expectancy-value perspective, conceptualized and operationalized with regard to the subject-specific characteristics of mathematics teacher training, in particular by distinguishing a mathematics mathematics-related attainment value as well as a mathematics teaching-related attainment value.

THEORETICAL FRAMEWORK

We draw on Eccles & Wigfield's (2020) situated expectancy-value theory as a theoretical framework and situate it in the field of university mathematics teacher education (in Germany) to better understand preservice teachers' intention to dropout or change their study program as well as their study satisfaction. By *dropout intention,* we mean preservice teachers' concrete thoughts or plans of quitting their university studies; by *program change intention,* their concrete thoughts or plans to change their study program, which for preservice teachers (in Germany) might mean to switch subjects (e.g., choosing another subject instead of mathematics but stay in a similar teacher training program) or choose another teacher training program (e.g., choosing a primary school teacher program instead of a secondary school teacher program). In contrast, *study satisfaction* is rather thought as an indicator of staying in the respective program (Geisler, Rach, et al., 2023).

Expectancy-value theory (Eccles & Wigfield, 2020) states that individuals' expectancy for success, subjective task values, and costs are important determinants for choosing tasks or activities as well as for performance and engagement in the chosen tasks and activities. *Expectancy* is defined as individuals' beliefs about how well they will do on an upcoming task (Eccles & Wigfield, 2020). Regarding a mathematics preservice teacher study program, expectancy can be seen as students' forecast of how well they can master the mathematical knowledge and skills that are taught at university. Expectancy has been shown to influence preservice teachers' dropout intention and study satisfaction (Geisler, Rach, et al., 2023; Geisler, Rolka, et al., 2023).

Values and costs are further subdivided, as described in the following. *Intrinsic value* is the anticipated enjoyment individuals expect to gain from doing a task (Eccles & Wigfield, 2020). Regarding a mathematics preservice teacher study program, this might mean that students enjoy working on the mathematical contents or tasks of their mathematics study. Intrinsic value influences preservice teachers' intention to dropout or change their study program as well as their study satisfaction (Geisler, Rach, et al., 2023; Geisler, Rolka, et al., 2023; Schnettler et al., 2020). *Attainment value* is the relative personal or identity-based importance attached by individuals to the task (Eccles & Wigfield, 2020). Regarding a mathematics preservice teacher study program, on the one hand this might mean that students value understanding mathematics contents as important for them (*mathematics-related attainment value*). On the other hand (and possibly simultaneously), this might mean that students value being able to teach mathematics contents well as essential for them (*teaching-related attainment value*; Gildehaus et al., 2023). Mathematics-related attainment value was shown to be associated with study dropout in former studies (Schnettler et al., 2020). For teaching-related attainment value, this question is still open. *Utility value* is the perceived usefulness of how well a task fits into an individual's plans (Eccles & Wigfield, 2020). Regarding a mathematics preservice teacher study program, this might mean that students value the university mathematics contents as relevant for their future profession as teachers. Such relevance appraisals were shown to be correlated with study satisfaction and program change intention (Eichler & Isaev, 2022).

Individuals will avoid tasks that *cost* too much relative to their benefits (Eccles & Wigfield, 2020). Regarding a mathematics preservice teacher study program, we consider emotional and effort costs to be particularly relevant (Gildehaus & Liebendörfer, 2021; Göller & Gildehaus, 2021). *Emotional cost* subsumes an individual's sense of potential negative psychological and emotional consequences associated with a task (Wigfield et al., 2017). Regarding studying mathematics, students' negative emotions such as frustration, helplessness (Göller & Gildehaus, 2021), fear of failure, or (performance) anxiety are emotional costs (Wigfield et al., 2017). *Effort cost* is the individual's sense about the amount of effort needed to complete a task and whether this perceived effort is worth it (Wigfield et al., 2017). Regarding a mathematics preservice teacher study program, this might refer to students' sense of how much effort is needed to complete their study successfully. Costs have been shown to predict students' dropout intention (Schnettler et al., 2020).

In line with Barron and Hulleman (2015), we conceptualize cost as a distinct component from values. More concretely, we assume that costs mediate the effects of expectancy and values on students' intention to dropout or change their study program as well as their study satisfaction, i.e., we assume that on the one hand e.g., dropout intention is affected by students' perceived cost (Schnettler et al., 2020), and on the other hand, students' sense of how costly their study feels is affected by their expectancy and values regarding their study program (cf. Figure 1).

RESEARCH AIM AND RESEARCH QUESTIONS

This study aims to better understand mathematics preservice teachers' intentions to dropout or change their study program and their satisfaction with their studies, particularly the role of students' expectancies, values, and costs therein. Following our theoretical considerations, we hypothesize that the effect of expectancy and values is mediated by students' perception of (effort and emotional) costs accompanying their mathematics study (cf. Figure 1). As these effects might differ for different groups and might be influenced by their prior mathematics knowledge, we aim to control for different study programs, genders, and high school grades in mathematics. Concretely, we investigate the following research question:

- How are preservice teachers' mathematics-specific expectancy, values, and costs related to their intention to dropout or change their study program as well as to their study satisfaction?

METHODS

In Autumn 2021, a total of $n = 209$ (154 females, 53 males, two without specification, $M_{age} = 19.9$) preservice teachers of two different study programs - 145 primary school preservice teachers (teaching pupils aged 6-10) and 64 secondary school preservice teachers (teaching pupils aged 10-19) - with mathematics as a study subject, took part in an online survey which was carried out in two first-semester lectures of a German university. Dropout intention, program change intention, and study satisfaction were measured by single items each (cf. Table 1). Expectancy was operationalized by a self-efficacy scale (Ramm et al., 2006). Values and cost were measured with a recently developed instrument (Gildehaus & Göller, 2023) which operationalizes intrinsic value, mathematics content-related attainment value, mathematics teaching-related attainment value, utility value, as well as effort and emotional cost (Table 1). We controlled for study program, gender, and high school grade in mathematics.

	Items	ω	Example
Dropout intention	1	-	I am seriously considering dropping out of my university studies.
Program change intention	1	-	I often think about changing my subject of study.
Study satisfaction	1	-	Overall, I am very satisfied with my studies so far.
Expectancy	3	.84	I am confident that I can master the mathematical skills that are taught
Intrinsic value	4	.89	I like mathematics at university
Attainment value mathematics-related	3	.91	It is important for me to understand mathematics content very well
Attainment value teaching-related	4	.92	For me, it is important to be able to explain mathematics well
Utility value	4	.90	The mathematical content studied is useful for my later career
Effort cost	3	.77	Sometimes, I am not sure if I have the energy to study mathematics successfully.
Emotional cost	3	.92	The mathematical contents of my studies depress me.
Study program	1	-	0 = primary, 1 = secondary school PST
Gender	1	-	1 = female, 2 = male
High school grade math	1	-	15 = best, 0 = poorest

Table 1: Overview of the analysed variables, numbers of items, and McDonald's ω.

A structural equation model was conducted for data analysis with dropout intention, program change intention, and study satisfaction as dependent (endogenous) variables using the R package lavaan (Rosseel, 2019). According to our theoretical considerations, we modelled costs as mediators and controlled for the study program, gender, and high school grade in mathematics. A schematic visualization of this structural equation model is given in Figure 1.

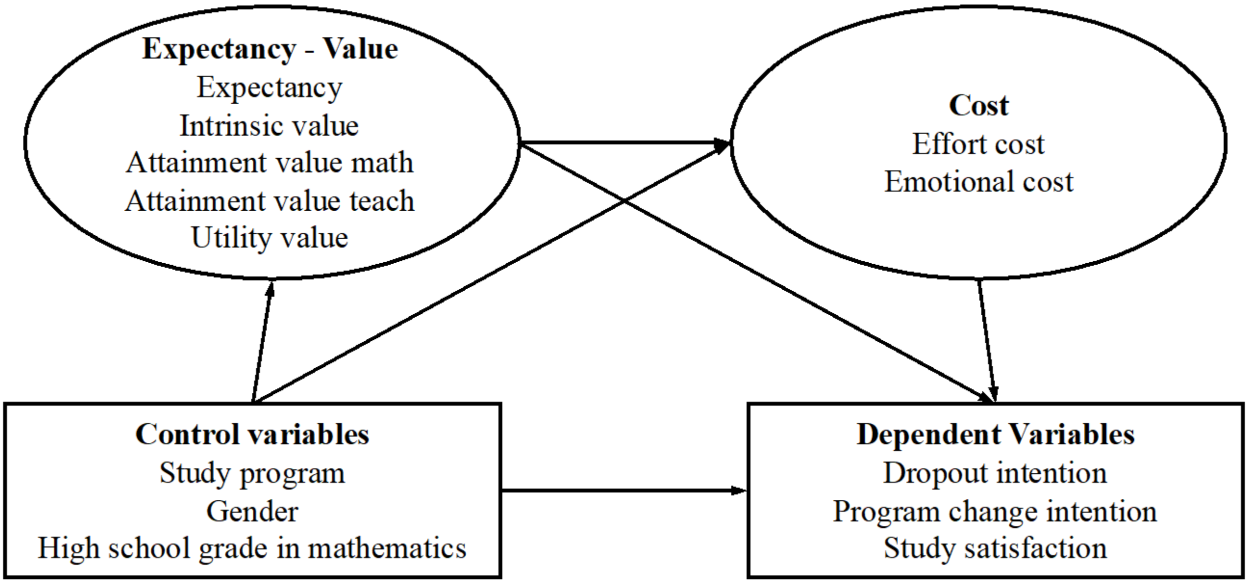

Figure 1: Schematic visualization of the structural equation model

RESULTS

The considered structural equation model has a satisfactory model fit (χ^2 = 447.15, df = 305, p < .001, CFI = .95, TLI = .94, RMSEA = .05, SRMR = .05). Figure 2 shows the significant standardized direct effects of the structural equation model.

Direct effects

In the considered model, mathematics preservice teachers with higher expectancy rather have a lower program change intention, higher study satisfaction, and lower effort and emotional costs. Students with higher emotional costs tend to have a higher dropout intention. Students with higher intrinsic value rather have lower emotional costs, students with higher teaching-related attainment value tend to have lower effort costs. There were no significant direct effects of values on dropout intention, program change intention, or study satisfaction (cf. Figure 2).

Mathematics secondary school preservice teachers rather intend to change their study program and tend to have lower study satisfaction than mathematics primary school preservice teachers. They tend to have a lower expectancy to master the mathematical knowledge and skills taught at university but higher intrinsic and mathematics-related attainment values than mathematics primary school preservice teachers. Students with better mathematics grades in high school rather intend to change their study program. No significant gender effects were found (cf. Figure 2).

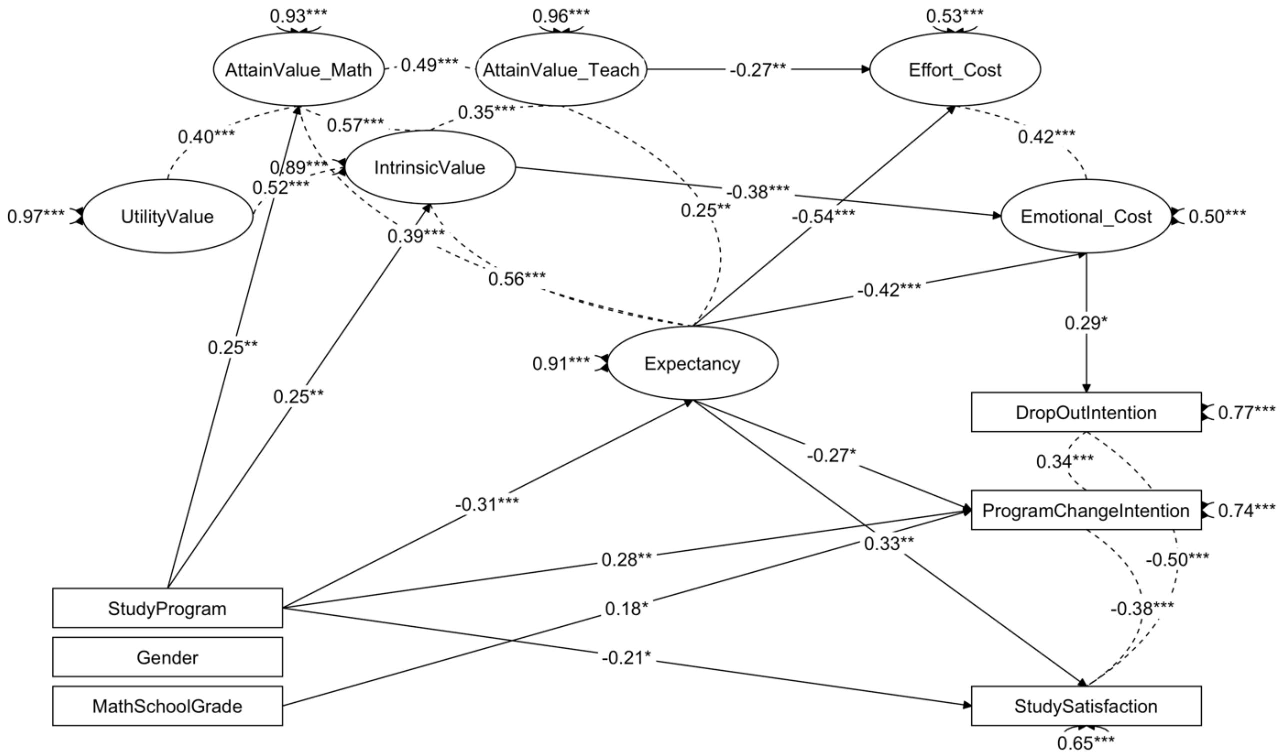

Figure 2: Visualization of the significant ($^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$) direct effects (standardized β) of the considered structural equation model.

Indirect and total effects

Expectancy has a significant indirect effect on dropout intention ($\beta = -0.12$, $p = .038$) fully mediated by emotional cost and a fully mediated significant total indirect effect on dropout intention via (effort and emotional) cost ($\beta = -0.17$, $p = .017$). The indirect effect of intrinsic value on dropout intention via emotional cost is not significant ($\beta = -0.11$, $p = .066$), like all other indirect effects of the analysed values.

Expectancy has significant total (direct and indirect via cost) effects on dropout intention ($\beta = -0.34$, $p = .002$), program change intention ($\beta = -0.30$, $p = .004$), and study satisfaction ($\beta = 0.37$, $p <. 001$). Utility value has a significant total effect on study satisfaction ($\beta = 0.18$, $p = .039$) and a slightly not significant total effect on dropout intention ($\beta = 0.18$, $p = .073$). Mathematics-related attainment value has a slightly not significant total effect on study satisfaction ($\beta = -0.19$, $p = .062$). All other total effects of the considered values are not significant ($p > .10$).

DISCUSSION

The results highlight the great importance of mathematics preservice teachers' expectancy to what degree they can master the mathematical knowledge and skills taught at university for their intention to dropout or change their study program, their study satisfaction, as well as their perceived effort and emotional cost. These results suggest that this subjective expectancy plays a more prominent role here than prior knowledge measured by the mathematics high school grade. However, other studies also document the importance of prior knowledge and achievement during the study (Geisler, Rach, et al., 2023; Geisler, Rolka, et al., 2023). Unlike findings from other studies (e.g., Geisler, Rolka, et al., 2023; Messerer et al., 2022; Schnettler et al., 2020), no significant effect of intrinsic and attainment values on dropout intention was found. However, intrinsic and teaching-related attainment values were significantly related to emotional and effort costs, respectively, indicating their importance for students' well-being. Although no direct effects of utility value were found, utility value had a total effect on study satisfaction, backing the correlation found by Eichler & Isaev (2022). The effect of cost on dropout intention was also found by Schnettler et al. (2020); beyond that, our study highlights the importance of emotional cost and the connections of costs to expectancy and values. This joint consideration of the links between all these variables situated in mathematics teacher education in one model is the strength of the present study.

Theoretically, the results imply that regarding costs as being affected by expectancy and values and as a mediator for the prediction of dropout intention may be considered further which might contribute to a better understanding of the relationship of expectancy, values, and costs. Practically, increasing students' expectancy that they will be able to master the mathematical knowledge and skills taught at university seems to be key when designing interventions for decreasing study dropout.

Limitations and Outlook

When interpreting the results, the comparatively small sample from only one university with its specific institutional context should be considered. Due to the study's cross-sectional design, all effects we report should be understood in a correlative and not a causal sense. Nevertheless, this study's results encourage further investigation of study dropout, program change intention, and study satisfaction designed for and situated in mathematics teacher education for a better theoretical understanding and to support students' valued participation in university mathematics to prevent study dropout.

References

Barron, K. E., & Hulleman, C. S. (2015). Expectancy-Value-Cost Model of Motivation. In *International Encyclopedia of the Social & Behavioral Sciences* (pp. 503–509). Elsevier.

Chen, X. (2013). *STEM Attrition: College Students' Paths Into and Out of STEM Fields* (NCES 2014-001). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Eccles, J. S., & Wigfield, A. (2020). From expectancy-value theory to situated expectancy-value theory: A developmental, social cognitive, and sociocultural perspective on motivation. *Contemporary Educational Psychology*, *61*, 101859.

Eichler, A., & Isaev, V. (2022). Improving Prospective Teachers' Beliefs About a Double Discontinuity Between School Mathematics and University Mathematics. *Journal Für Mathematik-Didaktik*. https://doi.org/10.1007/s13138-022-00206-w

Geisler, S., Rach, S., & Rolka, K. (2023). The relation between attitudes towards mathematics and dropout from university mathematics—The mediating role of satisfaction and achievement. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-022-10198-6

Geisler, S., Rolka, K., & Rach, S. (2023). Development of affect at the transition to university mathematics and its relation to dropout—Identifying related learning situations and deriving possible support measures. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-022-10200-1

Gildehaus, L., & Göller, R. (2023). Ein Fragebogen zur Erfassung fach- und berufsbezogener Werte für die Studienwahl von Mathematikstudierenden. *Beiträge zum Mathematikunterricht 2022*, 761–764. https://doi.org/10.17877/DE290R-23622

Gildehaus, L., & Liebendörfer, M. (2021). "I don't need this"—Understanding preservice teachers disaffection in mathematics. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 350–359). PME.

Gildehaus, L., Liebendörfer, M., & Schukajlow, S. (2023). Preservice teachers' mathematics-related values and expectancy in the transition from school to university. *Quadrante*, *32*(2), 25–48. https://doi.org/10.48489/quadrante.31191

Göller, R., & Gildehaus, L. (2021). Frustrated and helpless—Sources and consequences of students' negative deactivating emotions in university mathematics. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 321–328). PME.

Klemm, K. (2020). *Lehrkräftemangel in den MINT-Fächern: Kein Ende in Sicht*. https://www.telekom-stiftung.de/sites/default/files/mint-lehrkraeftebedarf-2020-ergebnisbericht.pdf

Messerer, L. A. S., Karst, K., & Janke, S. (2022). Choose wisely: Intrinsic motivation for enrollment is associated with ongoing intrinsic learning motivation, study success and dropout. *Studies in Higher Education*, 1–14. https://doi.org/10.1080/03075079.2022.2121814

Neugebauer, M., Heublein, U., & Daniel, A. (2019). Studienabbruch in Deutschland: Ausmaß, Ursachen, Folgen, Präventionsmöglichkeiten. *Zeitschrift für Erziehungswissenschaft*, *22*(5), 1025–1046. https://doi.org/10.1007/s11618-019-00904-1

Ramm, G., Adamsen, C., Neubrand, M., & Deutsches PISA-Konsortium (Eds.). (2006). *PISA 2003: Dokumentation der Erhebungsinstrumente*. Waxmann.

Rosseel, Y. (2019). *lavaan: Latent Variable Analysis*. https://cran.r-project.org/web/packages/lavaan/

Schnettler, T., Bobe, J., Scheunemann, A., Fries, S., & Grunschel, C. (2020). Is it still worth it? Applying expectancy-value theory to investigate the intraindividual motivational process of forming intentions to drop out from university. *Motivation and Emotion*, *44*(4), 491–507. https://doi.org/10.1007/s11031-020-09822-w

Wigfield, A., Rosenzweig, E., & Eccles, J. (2017). Achievement values. In *Handbook of competence and motivation: Theory and application* (pp. 116–134). Guilford Press New York, NY.

STUDENTS' CHANGING METARULES DURING AND AFTER WATCHING DIALOGIC INSTRUCTIONAL VIDEOS

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Dialogic instructional videos feature authentic conversations of students as they engage in complex mathematical problems. Because these videos show students engaging in rich mathematical interactions students might use them as models for how they should engage in such interactions. In this study, we investigated how watching a dialogic video that showed two students creating pictures to illustrate mathematical relationships shaped what two pairs of students thought was necessary to include in their own pictures. We found that while the video the students watched did indeed shape what they thought was necessary to include in their pictures, the degree to which they felt they needed to mirror the pictures in the video varied considerably.

INTRODUCTION AND PERSPECTIVE

Instructional videos are appealing because they can flexibly offer additional instruction. Students can view them on their own time at their own pace or instructors can integrate them directly into classroom instruction. While the convenience and access to additional instruction that videos provide is compelling, educators should critically reflect on the quality of that instruction. Many instructional videos feature an expert explaining a concept or procedure (Bowers et al., 2012), essentially providing a lecture experience. However, a meta-analysis of classroom studies comparing lecture to alternatives suggests that the alternatives can be more productive (Freeman et al., 2014).

One way video creators have begun to go beyond recreating lecture on video is to create dialogic videos, those that feature the authentic dialogue of students as they engage with complex mathematical problems (e.g., Lobato et al., 2019). These videos have great potential because they allow students to indirectly participate in negotiating mathematical meanings, evaluating and critiquing the reasoning of others, and comparing peers' ways of reasoning to their own (Lobato et al., 2023). These practices mirror the types of rich interactions researchers and educators advocate for in classroom settings (National Council of Teachers of Mathematics, 2014). To investigate how viewing these videos shaped students' ways of interacting, we adopted a commognitive perspective (Sfard, 2008).

The commognitive perspective asserts that thinking is "an individualized version of interpersonal communicat[ion]" (Sfard, 2008, p. 81). Thus, instead of conceiving of learning as the acquisition of concepts, skills, and procedures, learning is defined as being able to participate in an expanding set of discourses. This requires learning the rules of these discourses.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 248–255). PME.

Sfard suggests that students learn two types of rules: object-level rules and meta-level rules. In general, "object-level rules are narratives about regularities in the behaviour of objects of the discourse, whereas meta-level narratives or meta rules define patterns in the activity of the learners trying to produce and substantiate object-level narratives." (Sfard, 2008, p. 204). For example, 2+3 = 5 is an object-level rule in arithmetic because it is a narrative about the relationship between the objects 2, 3, and 5. However, the rule "You can add the addends in either order (e.g., 2+3 or 3+2)" is meta-level because it governs how to produce object-level rules. Meta-level rules help us substantiate our claims. As such, the development of meta rules is important because they can help us to "become aware of new possibilities and arrive at a new vision of things" (Sfard, 2007, p.577).

While meta rules can seem firm because they govern how to endorse object-level narratives, they can change over time. This is because they are a result of patterned activity among a community's interlocutors. In this way, they are a product of, often tacit, social negotiation. This means the rules themselves are often tacit. However, this is not always the case. At times, participants in the community will make explicit the rules for arriving at object-level narratives. For this reason, Sfard distinguishes between *enacted* meta rules, the rules that seem to be governing interlocutors' actual behaviour, and *endorsed* meta rules, those that are explicitly stated as rules and agreed upon by the community members.

Our research seeks to provide insights into how viewing dialogic videos might shape students' development of meta rules. This led to the following research question, "As secondary students solve tasks and view dialogic videos of students solving similar tasks, what meta rules were developed and what changes occurred after watching the videos?"

METHODS

Data were collected through four one-hour semi-structured interviews with four pairs of secondary students (grades 11 and 12). Conducting interviews in pairs allowed for meaningful interactions between students, not just student and interviewer. During the sessions, the pairs were tasked with solving a problem, followed by questions about their thought processes. They then watched a segment of a dialogic video featuring two students, Josh and Arobindo, solving a similar problem, and were given the option to revise their work.

The video the students watched showed Josh and Arobindo in the bottom right corner of the screen, with their work in the upper left. Viewers could choose to show or hide captions (see Figure 1). While the teacher's voice can be heard as he assigned tasks and prompted explanations from Josh and Arobindo about their solution paths, mathematical reasoning, and how they showed mathematical relationships, his presence was not visible. The video unit was on exponential functions, totalling 34 videos across 7 lessons. Throughout the unit Josh and Arobindo explored the exponential growth of magical beanstalks.

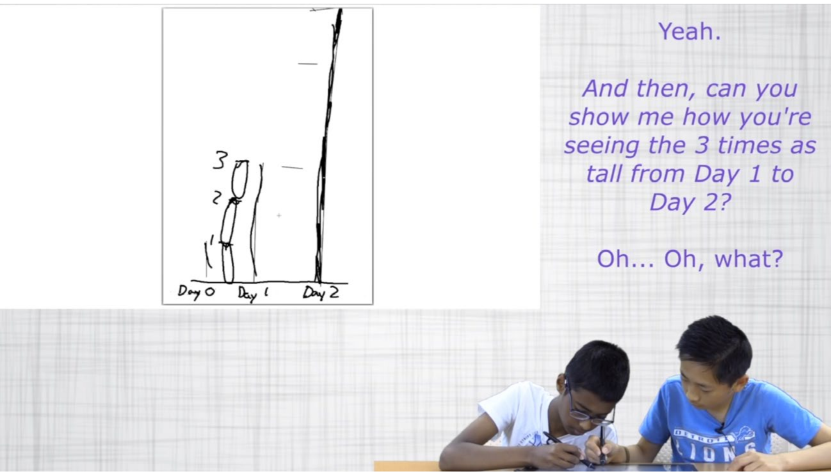

Figure 1. A screenshot of the video the interviewees watched during their interview.

Data analysis proceeded with us first creating descriptive accounts of the videotaped interviews. After reviewing across the accounts, we chose to focus our analysis on two of the pairs' responses to the first task we posed in the interview (Figure 2). This task was selected because the participants seemed to actively negotiate the meta rules for the interview session. Our focus on these two pairs is driven by the observation that the pairs reflect ends on a spectrum of how these videos can influence meta rule development.

Task 1: Consider a beanstalk whose height on day 0 is 1 cm and whose height triples each day. Draw a picture of the height of the beanstalk on Day 0, Day 1, and Day 2 that shows the tripling from one day to the next. Can you mark in any math relationships you see?

Figure 2. The interview task.

We began analysis by perusing the descriptive accounts to generate hypotheses about the students' meta rules. We focused on what seemed to count for the students as showing mathematical relationships, including showing the tripling in their pictures. The initial hypotheses were further refined by re-watching the videos and generating transcripts. Once we felt confident that we had inferred meta rules consistent with the students' actions and dialogue, we looked for changes in their meta rules before and after viewing the instructional video. Finally, we examined how those changes were related to the actions of dialogue of the students in video, Josh and Arobindo.

FINDINGS: METARULES BEFORE AND AFTER

Our findings suggest that the video shaped the development of meta rules for both pairs of students, but in different ways. For Celina and Olympe the video seemed to make them more confident in their initial idea that drawing a graph or writing an equation counted as "showing a mathematical relationship." This is because Celina felt that her initial drawing showed the same relationships as Josh and Arobindo's and was thus sufficient. On the other hand, Olympe recognized that her drawing was quite different from Josh and Arobindo's, but she felt that it was still sufficient because it made sense to her and she understood what Josh and Arobindo were saying. In contrast, Daniel and Peter wanted to change their picture after watching the video. They originally focused on using their picture to find the height of the beanstalk on Day 2. However, after watching the video, they revised their picture to be more similar to Josh and Arobindo's. We provide more detail about our analysis that supports these claims below.

Olympe and Celina

Alicia, the interviewer for Olympe and Celina, asked them to engage in the task by saying, "I'd like for you to talk to each other…yeah… let me know when you're done and I'll ask you questions about your work." Olympe began by inquiring of Celina, "What do you want to do?" Celina responded, "So, should we…" as she drew a long vertical line on the paper. Olympe responded with, "Wait, can we do, erm…, can we…?" and drew the axes to a graph. Alicia replied, "You can do whatever you want." Celina then decided to continue with her line, while Olympe decided to create a graph. After she finished her graph, Olympe asked Alicia, "Does that work?" Alicia replied, "Yeah, yeah, yeah, it's up to you, like I said, no right or wrong answers; just wanted to see how you're thinking about it."

In the above exchange we see evidence for the metarule (MR) that Olympe and Celina first appeared to operate with, *MR 1: The objective of this task is to show our thinking, but we're not sure what counts and we seek approval.* Olympe and Celina grappled with uncertainty about the task, negotiating their drawings and settled on different pictures (Figure 3). Olympe's seeking approval from Alicia suggested a perceived need for approval of the solution path, yet the specific requirements remained unclear. This reinforced the idea of seeking approval for acceptability. Olympe continued with her drawing, seeking Alicia's approval once more, with Alicia reiterating her freedom of representation.

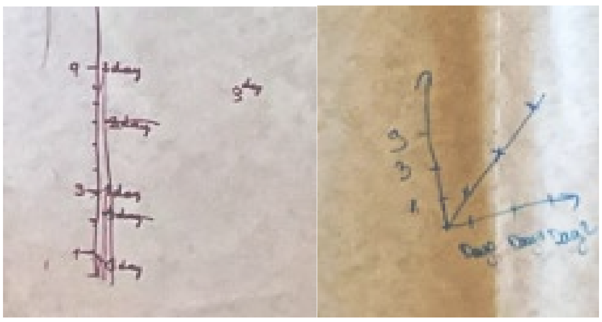

Figure 3. Celina's picture on left, Olympe's on right

After she had finished her drawing, Celina re-read the task statement, which asked them to mark in the math relationships they saw. Alicia then asked them to explain those relationships, but backtracked as she realized they were still grappling with the question. Celina said to Olympe, "Okay, we're not done, mark in math relationships that you see. You just drew something. That's not a math relationship." Olympe responded by trying to find an equation. She said, "Well I put x at the third or whatever, and that wouldn't work because one, one exponential three, would be one, so that doesn't work, oh wait, no…" Celina then joined her in trying to find an equation. Eventually, Alicia said "You don't have to put it into an equation, we want to see sort of what your pictures sort of look like. But it's fine if you want to." Celina stopped, and said "Well, yeah, that's my picture then, I guess (see Figure 3)."

During this exchange we believe Olympe was initially operating under the meta rule, *MR 2: Drawing a graph is acceptable as a solution.* However, it seems Celina was operating under a different meta rule, *MR 3: Drawing a graph is does not count as marking in mathematical relationships.* Together, they seemed to develop a new meta rule, *MR 4: An equation counts as a math relationship*. Olympe had stopped writing before Celina's comment, "Okay we're not done, mark in math relationships that you see. You just drew something. That's not a math relationship." This suggests that Olympe thought her graph was sufficient (MR 2), while Celina did not (MR 3). Olympe then started to create an equation (MR 4) as she said, "Well I put x at the third whatever, and that wouldn't work because one, one exponential three, would be one, so that doesn't work, oh wait, no…" This meta rule may have been stunted as Alicia again suggested that she does not have to put it into an equation and a drawing is sufficient.

Olympe and Celina then watched a clip from a dialogic instructional video that showed Josh and Arobindo drawing a picture that showed the height of the beanstalk on Day 0, Day 1, and Day 2 and illustrated mathematical relationships. They represented the height of the beanstalk with vertical lines and showed mathematical relationships by drawing ovals next to those lines. Specifically, they showed that the height increased by a factor of three from Day 0 to Day 1 by drawing three ovals that were each the same height as the vertical line representing Day 0 next to the line representing Day 1 (see the screenshot in Figure 1). Similarly, they drew three large ovals, each with three smaller ovals inside them, next to the line representing Day 2. This showed that the Day 2 height was equivalent to 3 groups of 3 copies of the Day 0 height. Notably, these drawings were the result of some negotiation with the instructor (John) around what counted as "showing a mathematical relationship."

After Olympe and Celina summarized what happened in the video in their own words, Alicia asked, "How does your picture compare to theirs?" Celina responded, "I think that mine is pretty similar." In contrast, Olympe said, "I think mine is pretty far away." Alicia then asked if they thought their pictures showed the same relationships, and they both responded "Yeah." Alicia then asked if they wanted to change their picture, but neither did. Olympe said, "Well my drawing makes sense to me, but I probably couldn't explain it to someone. So, if I had to teach it to someone else I would probably use that [Josh and Arobindo's picture] because it's very clear. But in my head, it's very clear." While Celina responded, "Yeah I think I would probably use that one [Josh and Arobindo's], but I think mine is understandable."

After watching the video of Josh and Arobindo, Celina and Olympe's *MR1* seemed to change to *MR 5: The purpose of the task is to explain Josh and Arobindo's reasoning and compare our drawing to theirs. As such, we don't need approval anymore.* Furthermore, Celina seemed to develop a new metarule, *MR 6: My quantitative explanation was sufficient and showed the same relationships as Josh and Arobindo's* as did Olympe, with *MR 7: My drawing is a useful tool for my thinking, but for explaining to someone else, Josh and Arobindo's explanation is clearer.* Evidence for MR 6 includes Celina's statement "mine is pretty similar" and evidence for MR 7 includes Olympe choosing not to revise her picture and stating her thinking was "very clear."

Daniel and Peter

John interviewed Daniel and Peter. After he posed Task 1, Daniel and Peter worked together to draw the picture shown in see Figure 4. Peter, after asking Daniel if they "should also label the height," then asked, "wouldn't it be nine?" With some reassurance from John of "you're doing great, you're doing great," they continued with Peter asking Daniel "would you cube it to triple it?"

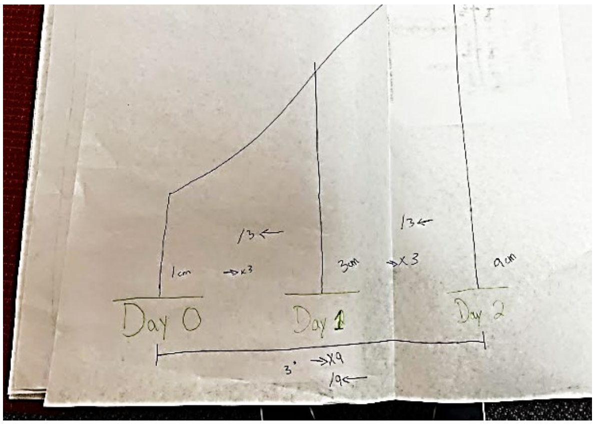

Figure 4. Daniel and Peter's initial picture

The first rule Daniel and Peter seemed to operate under was *MR 1: The purpose of engaging in this task is to solve the task accurately.* Because Daniel asked Peter "wouldn't it be nine?" and Daniel responded hesitantly with "yeah?," we inferred that the two interviewees were taking seriously the task of finding mathematical relationships, which supports our interpretation for MR 1.

After creating a line graph showing the beanstalk's height Days 0, 1, and 2, they began expressing and illustrating math relationships they saw by writing "x3" between the days. Daniel explained, "From each day, there's a multiple of three. After one day, after another day passes, it's a multiple three, so it increases by that much." John then asked for other math relationships, telling them the first one they found was "a great one." Daniel noticed that one could also say that, going in the opposite direction, the height was by divided by three over each day and wrote "/3" between the lines representing the beanstalk's heights. Peter then questioned if exponential growth qualifies as a relationship, showing uncertainty about the criteria.

In this episode, Daniel and Peter provided evidence they were operating under *MR 2: We think that a graph annotated with multiplication and division symbols counts as showing mathematical relationships, but we're not sure.* At this point the students had drawn a graph and marked in the factors by which the height changed. After some exchange, Daniel and Peter appeared to feel satisfied with their work, with Daniel indicating they were finished by saying, "Okay."

Daniel and Peter then watched the same clip that Olympe and Celina watched of Josh and Arobindo illustrating mathematical relationships by drawing sets of ovals. John then asked Daniel and Peter to explain what Josh and Arobindo's drawing showed. They pointed out the Josh and Arobindo showed the tripling from one day to the next with the ovals that they drew. They explained that the ovals showed, as Daniel put it, "how each segment of the previous day is built within the next height." Peter then elaborated, explaining that the ovals were showing the tripling from one day to the next. Daniel and Peter were then asked to redo the task and compare the picture they drew with the picture of Josh and Arobindo's. They redrew the three vertical lines representing the heights on Days 0, 1, and 2, similar to what they had drawn before, but this time they annotated the Day 1 picture with three segments to the side and the Day 2 picture with nine segments to the side. These segments seemed to serve the same purpose as Josh and Arobindo's ovals as Peter explained how they showed the tripling from one day to the next. In fact, when asked to compare, Peter made the connection explicit saying they were "like the ovals."

From their response we inferred they had developed two new meta rules, *MR 3: Josh and Arobindo's drawing, particularly the subdivision of the heights on each day, is an acceptable way to show the tripling relationship* and *MR 4: Our drawing should be more similar to Josh and Arobindo's.* We infer these meta rules from the fact that they revised their picture to show the same relationships that Josh and Arobindo showed.

DISCUSSION

The dialogic instructional video the interviewees watched featured the authentic dialogue of two students as they worked together to draw a picture illustrating mathematical relationships. We hypothesized that having students watch videos that showed an example of creating a picture that showed mathematical relationships would shape the development of their own meta rules regarding how to communicate mathematical relationships. Our findings illustrate that our hypothesis was correct, though the meta rules the two pairs developed were quite different. Daniel and Peter developed meta rules that suggested they draw pictures that mirrored Josh and Arobindo's, ones that showed similar relationships in similar ways. In contrast, the video seemed to give Olympe and Celina confidence that their original pictures were sufficient. Since their original thinking was broadly consistent with Josh and Arobindo, they did not feel the need to revise their pictures to look like Josh and Arobindo's. This may be related to what they saw as the meta rules related to the purpose of the task, as

Olympe articulated that if she had needed to explain to another student, she might use Josh and Arobindo's representation.

These results suggest that dialogic instructional videos shape the development of students' meta rules. Both pairs of students seemed to attend to the videos and use Josh and Arobindo's work as a cue for what type of picture and explanation satisfied the task requirements. However, if teachers want to use dialogic videos to develop particular meta rules or establish particular expectations for drawings or explanations, they should be aware that they will need to go beyond simply showing the videos. This could include being explicit about what they found productive about the pictures and explanations featured in the videos. Similarly, they may want to consider intentionally eliciting and responding to students' developing meta rules.

REFERENCES

Bowers, J., Passentino, G., & Connors, C. (2012). What is the complement to a procedural video? *Journal of Computers in Mathematics and Science Teaching, 31*(3), 213−248.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings for the National Academy of Sciences of the United States of America* (vol. 111(no. 23, pp. 8410−8415). https://doi.org/10.1073/pnas.1319030111.

Lobato, J., Gruver, J., Foster, M. (2023). Students' development of mathematical meanings while participating vicariously in conversations between other students in instructional videos. *Journal of Mathematical Behavior, 71.*

Lobato, J., Walters, C. D., Walker, C., & Voigt, M. (2019). How do learners approach dialogic, on-line mathematics videos. *Digital Experiences in Mathematics Education, 5*(1), 1−35. https://doi.org/10.1007/s40751-018-0043-6

National Council of Teachers of Mathematics (2014). *Principles to actions: Ensuring mathematical success for all*. Author.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, *16*(4), 565-613.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge university press.

USING APPLICATIONS IN FIRST-SEMESTER CALCULUS FOR ENGINEERING. SOURCES OF APPLICATIONS, USE OF TEXTBOOKS, AND EXTERNAL CONSTRAINTS.

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In this paper, we analyse the use of applications by instructors with different backgrounds teaching first-semester calculus in engineering programmes. Adopting the perspective of the Anthropological Theory of the Didactic (ATD), we investigate the teachers' sources of these applications, as well as the teachers' rationales for using or not using them. Our results indicate that while teachers may draw on their professional experience as a source of real-world applications, some opt to adhere to examples provided in the course textbook. Moreover, other constraints, such as the perceived size of the syllabus, the heterogeneity of classes, and the students' lack of advanced knowledge may hinder teachers' use of applications.

INTRODUCTION

The teaching of mathematics to engineering students has been a topic of interest at conferences and among researchers in mathematics education for more than a century. Slaught (1908) writes that at the Chicago Symposium on Mathematics for Engineering Students in December 1907, concern was expressed over the need to anchor mathematics courses to concrete applications:

> There is no better place than the mathematics classroom to develop logical thinking, but this must be done in connection with well-selected concrete problems, and not in the domain of abstract and theoretical considerations (Slaught, 1908, p. 281).

Although this issue dates to the beginning of the last century, it remains a concern in engineering education today. In their pioneering work, Kent and Noss (2003) interviewed mathematicians who teach mathematics in engineering programmes, as well as engineers who teach other courses. The authors point out that although mathematics are an important element of engineers' education, a fundamental issue concerns the type of mathematics needed and the point at which they should be taught within an engineering programme; they also note the importance of modelling and applications in these programmes. In this vein, López-Díaz and Peña (2022) claim that in the first year of a STEM degree it is essential to engage the class by teaching mathematics using real-world problems, since this can help students develop mathematical thinking for practical applications and strengthen their ability to draw connections between mathematics and the engineering disciplines. The mathematics taught in engineering programmes have been described as disconnected from the practices of professional engineers. For instance, Quéré (2017) sent an anonymous online questionnaire to practicing French engineers to ascertain their mathematical

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 256–263). PME.

needs in the workplace. Only 47% of the 267 respondents declared that they have a real need for university mathematics in their daily work, and their most frequent use of mathematics involved simulations, modelling, data analysis, and calculations.

More recently, with respect to the training of engineers, Peters (2022) and Faulkner et al. (2020) also advocate for a teaching of mathematics that is more integrated and contextualized to the needs of engineering students. Complementing this argument, Castela and Romo-Vázquez (2022) argue that there is a need for more research into mathematics used in the workplace, in order to provide mathematics teachers with a better understanding of the professional contexts (engineering in particular) that use mathematics in non-academic ways.

We are interested in exploring the use of applications in the teaching of mathematics to engineering students, with a particular focus on calculus courses. Teachers in engineering programmes usually come from a variety of professional and academic backgrounds, yet there is scarce research on how their backgrounds influence their teaching practices. Our study focusing on two teachers with different backgrounds and professional experience (González-Martín & Hernandes-Gomes, 2020) suggests that this experience may provide justifications for their teaching practices and use of mathematics. In another project (González-Martín & Hernandes-Gomes, 2021) we went further, analysing a case of teachers teaching the same content. We saw indications that teachers with different backgrounds may use different applications when teaching calculus, with varying degrees of real-world usefulness. We dive deeper into this issue in the present study, investigating what types of applications are used by calculus teachers with different academic and professional backgrounds in engineering programmes, as well as how the teachers' professional and academic backgrounds influence their use of these applications.

THEORETICAL FRAMEWORK

Our study focuses on the practices employed by teachers as they prepare and deliver calculus courses for engineering students, and how these teachers select and use applications. It is appropriate to take an anthropological approach to better understand the reasons that support these practices. We therefore use Chevallard's (1999) Anthropological Theory of the Didactic (ATD).

Praxeologies are a key notion of ATD that allows for the modelling of human activity. A praxeology is composed of four components: the *types of tasks* to be carried out, a *technique* that allows these tasks to be completed, a *rationale* (or *technology*) that explains and justifies the technique, and a *theory* that explains and justifies the rationales. One important principle of ATD is that institutions influence and put constraints on the learning that happens within them, which has an impact on individuals operating within the institution. Furthermore, the position that an individual occupies in an institution also influences their behaviour. For instance, in a secondary mathematics class, when addressing functions, the teacher and students perform different tasks and have different responsibilities because they occupy different positions. In our case, we look at teachers who currently occupy the same position (*teaching a calculus course in an engineering programme*), but who have previously occupied different positions in other institutions. Consequently, they may approach the tasks related to preparing and teaching a calculus course in different ways.

This large task can be subdivided into various sub-tasks, each with its own technique. One sub-task involves preparing and using practical applications to supplement the course content, and we are interested in analysing how this sub-task is carried out by teachers with different backgrounds. The lens offered by ATD allows us to explore the techniques used to accomplish this sub-task; more specifically, we identify and investigate the differences in the teachers' practices, pinpoint the rationales behind the teachers' choices, and determine whether these rationales are influenced by the teachers' previous experience in other institutions.

METHODS

In May 2023, we interviewed six calculus teachers on faculty in engineering programmes at a private university in São Paulo, Brazil. Each teacher has a different academic background (see Figure 1). All interviews took place in Portuguese and lasted between 60 and 90 minutes. They were recorded and transcribed, with excerpts translated into English for this paper. The interview questionnaire was comprised of four parts: 1) demographic questions, which aimed to obtain information about each teacher's academic and professional background; 2) general questions about how the teachers prepare for their calculus courses; 3) questions about the applications the teachers use involving limits, derivatives and integrals; and, 4) questions on specific exercises from the class textbook.

T1 Male	**T2** Male	**T3** Female	**T4** Male	**T5** Female	**T6** Female
Mathematics (B)	Physics (B)	Physical & Biomol. Sciences (B)	Mathematics (B)	Mathematics (B)	Mathematics (B)
Mathematics Education (M)	Physics (M)	Biomolecular Physics (M)	Applied Mathematics (M)	Administration (M)	Space Engineering and Technology (M)
Mathematics Education (D)	Materials Engineering (D)(IP)	Mechanical Engineering (D)	Mathematics (D) (NF)	Administration (D)(IP)	Mechanical Engineering (D)

Figure 1: Profile of the six university teachers (B: bachelor, M: master, D: doctorate, IP: in progress; NF: not finished)

Our analyses are ongoing. For this paper, we focus on the interviews with teachers T4, T5 and T6, because although all three teachers have a bachelor's degree in mathematics, they each have a different graduate degree, which is more likely to result in clear dissimilarities in their practices. We previously provided preliminary data from T1 and T3 in González-Martín & Hernandes-Gomes (submitted).

T4, T5, and T6 teach first-year calculus to engineering students. This course (Calculus I) is included in the Basic Curricular Components, a set of courses common to all engineering programmes at the teachers' university. Calculus I is a semester-long course that covers functions, limits, and derivatives, and concludes with rate of change and optimisation problems. T4 has been teaching at the university level since 2000 (24 years) and has taught calculus courses throughout his career. He also worked at a bank for three years and as a programmer for two years, which is what led him to study mathematics. T5 has been a university teacher since 2018 (six years) and has taught calculus for four years. She has eight years of professional experience as a systems analyst and six years of experience in analytics, working on big data analysis and building statistical models for fraud prevention and credit granting purposes. T6 has been teaching at the university level since 2001 (23 years) and has taught calculus courses since 2003. Although she does not have professional experience outside university, she has taught other courses in engineering (e.g., operational research, software for production engineering), in addition to supervising a number of production engineering students' capstone projects.

After the interviews were transcribed, the teachers' responses were coded in terms of tasks, techniques, and rationales, allowing us to organise the data for analysis. We paid special attention to the presence of applications in the teacher's practices, the repertoire of applications at their disposal and their knowledge of the applications they use, the reasons behind their decision to use (or avoid) these applications, and the difficulties they face in selecting and using applications. Finally, we also attempted to connect these issues and the teachers' choices to their training and professional experience.

DATA ANALYSIS

All three participants see their background as a major influence on their approach to preparing a calculus course for engineers (Figure 2).

T4	Practice (Pr.)	"In my administration courses, when I teach the chain rule, I don't give a function composed with another function, composed with another function… There's no need. However, here in engineering, I do give [the students] a function composed with another function, composed with another function, composed with another function. I try to strike a balance, based on the types of students in my class."
	Rationale (Rat.) Source (So.)	"I know because I did applied mathematics […] and I had to work with differential equations. […] I know that to solve all that, I had to use a lot of maths and I needed complex stuff. I didn't have to deal with easy derivatives like those in the Calc I book."
T5	Pr.	"I try to take [the students] beyond the textbook, since I notice that both in calculus and in statistics […], how does that course connect with the real world?"
	So.	"I think that having professional experience, mainly in statistics, offers me this possibility of bringing that experience to Calculus."
T6	Pr.	[After citing some examples of applications] "How can we leverage our experience so [students] can also be exposed to a bit of it, together with maths? […] It's a bit of this experience that I often turn to, at appropriate moments, which makes the class more interesting."
	So.	"I think that my training has helped me develop this ease to recognize applications in engineering."

Figure 2: The participants' views on their background's influence on their teaching practices.

All participants point to the fact that they are teaching future engineers as a rationale for using applications; they feel the need to relate the course content to real-life situations. We also observe that T4 uses part of his experience (dealing with complicated differential equations) as a rationale for tackling demanding calculations and extensive techniques. In what follows, we provide an overview of how the participants materialise this rationale (they teach engineering students, therefore connections to real-world scenarios are important) when teaching specific calculus content. Figure 3 summarises the participants' responses regarding limits.

T4	Pr. So.	"I provide a very simple model, which is the timing of parking [meters…]. What happens at the top of each hour? Then, when you study continuity, it has to do with limits. […] I'd like a lot, but I don't do it, to show examples with pollution in water, which is related to cost. And the purer you want the water, the more the cost increases and that tends to infinity. […] I have many examples of applications like that, but we have that issue of time and the syllabus."
	Rat.	"We are in the first semester, and we end up prioritising procedures. […] We have a huge syllabus […] we have to prioritise more key points. […] If I showed several applied examples of limits in engineering […], what's the cost? The cost may be that I don't have time for the techniques in derivatives."
T5	Pr.	"I present some very superficial examples […] Finally, I end up saying '[…] besides those simple and silly applications, derivatives come from limits.' And then, when we get to derivatives, I get to show some applications that they may grasp."
	Rat.	"Limits, for me… they need to improve, it's very poor. […] It's quite complex, you can't even set an example."
T6	Pr.	"I usually tell them: 'imagine we have a system, and it depends on a certain value, let it be tension, or current, and then I have to get very close to that, without overtaking since then the system could be unstable'. I use that intuitive idea a bit, but it is not a very developed application."
	Rat. So.	"Engineers need that interpretation a lot, not just algebra. […] This comes from my own training. […] I saw a lot of that in electronics, in general electricity, electromagnetism, part of microwaves, there're a lot of things. […] You need to analyse […] if the system is stable or unstable."

Figure 3: The participants' use of applications in limits.

We can see that T4 believes the density of the syllabus places constraints on the number of practical examples that he can share from his own experience. At several points during the interview, he mentions the pressure to cover all the content in the syllabus, which is his rationale for not exploring applications in depth. In a similar vein, due to the amount of content she must cover, T5 seems to see limits as a necessary evil to be addressed before proceeding to derivatives. T6 manages to provide students with a few applications that seem intended to distract from the number of algebraic procedures present in the chapter on limits. We can also see how her experience leads her to believe in the importance of providing examples to her students. In Figure 4 we summarise the participants' responses concerning derivatives.

T4	Pr.	"I start with a profit function. It's important to know if your profit is increasing or decreasing. […] You make a graph, and we observe it. It's increasing, your profit is increasing here. […] But how do you know at a point? I start with the tangent line. […] In engineering, instead of profit, I use a distance function […] and talk about speed."
	So.	"Those applications are drawn from [various] books that I work with."
T5	Pr.	"We start with tangent lines, normal lines, this is physics, used in optics: reflexion, refraction… […]. Then we use the first derivative to see growth; then, second derivative to see concavity. Then, this application is a bit fake, since it is an application of calculus [used to solve more] calculus. […] But after they see in physics… I have a distance as a function of time, you differentiate and get the speed, differentiate and get the acceleration. […] This semester, since I was in chemical engineering, I got an example from chemistry, from [the book Stewart]."
	So.	"These applications come from the physics book. […] Let me make a confession. Optimisation problems, I get very uncomfortable, since they are like about spherical cows in a vacuum. It's a thing that is not real, there's a lot like that in the book. […] It's different than statistics, when I give an example [derived from statistics], it makes sense. Ok, let's study the average consumption of vehicles."
T6	Pr. Rat.	"When I tell students that the next concept is in the part about optimization […] you use applications from physics. […] And our syllabus is coordinated with that of physics. Then, they see it in physics and in calculus. […] [regarding the use of realistic applications], there's quite some distance [from reality].

		What happens is that, to do a realistic application, we'd have to go deeper. We'd need to have more physical, mechanical tools. To be able to talk about electricity, for instance […] why do they study it much later? [to support the idea that students need more content]"
	Rat. So.	"Groups are heterogeneous now, they're not only production or only mechanical [engineers] […] This makes things harder. […] The applications are drawn a bit from my training, from my experience, from the book."

Figure 4: The participants' use of applications in derivatives.

T4 and T5 rely mostly on textbooks for applications, drawing less on their experience. This seems consistent with the study by Mesa and Griffiths (2012), which shows the influence of textbooks at the university level. It is possible that the perceived importance of textbooks puts pressure on teachers to use them. In the case of T5, it is also possible that because her professional experience was more centred around statistics, this may explain her hesitancy to provide applications other than those present in the textbooks she uses. As for T6, her experience with electricity had little to do with derivatives, which could also explain her reliance on the textbook. We also identify two rationales related to organisational decisions made by the faculty of engineering: the students represent a mix of engineering specialties, and the teachers are reluctant to present applications that may interest only a portion of the class. In addition, because calculus is a first-semester course, T5 feels that she cannot use many real-world applications, given the students' lack of advanced knowledge. In Figure 5, we summarise the participants' responses concerning integrals.

T4	Pr.	"I basically teach techniques of integration, calculation of areas, improper integrals, arc length."
	Rat.	"I think they see [applications] in physics."
T5	Pr. So.	"The fact that it is an area. […] In some exercises, we do the rotation of the axes, and there's an exercise where I demonstrate the volume of a sphere. […] The strongest application that I mention […] is probability, which is my area of knowledge. Then, it's quite easy to talk about probability […] and there are many examples, many estimations […] and you are constantly calculating integrals. […] For them, it's okay. I don't think that there is much diversification in terms of applications of integrals."
	So.	The book is a resource for applications, and she draws on her experience when talking about probability.
T6	Pr.	"In integrals, I talk a bit about what I know concerning electronics. There are many [applications] I can use, but in this course, we mostly calculate arc length, volume, areas…"
	So.	"The other day, a production engineer sent me the design for a company's packaging […] They needed to calculate the maximum volume for a certain product. […] [She] just had to calculate the integrals, then add everything and you get the total volume. […] I mean that, in practice, they also use [integrals]." "Many [applications] come from the book. I remember a few applications in electricity, right? But many come from books and physics."

Figure 5: The participants' use of applications in integrals.

T4 uses another rationale to explain why he does not use applications when teaching the chapter on integrals: students will encounter these applications in another course (physics). This rationale is likely connected to his previous one about the density of the syllabus and the lack of time to cover the content. Regarding T5, although she seems to rely on the textbook here too, she manages to draw connections to her background and experience. Finally, T6 seems to balance her use of the textbook with examples pulled from her own experience working on electrical engineering projects.

FINAL CONSIDERATIONS

While some of our results are fairly predictable, our study also reveals important phenomena. As expected, and as observed in González-Martín & Hernandes-Gomes (2021, submitted), teachers' academic and professional backgrounds can provide a rich source of material to enhance their teaching in the context of an engineering programme. However, building on the results of our previous publications, it seems that certain profiles are less apt to yield realistic, engaging applications, as seems to be the case with T4 and T5. Although all three teachers believe it is important to use applications in their courses, they employ different techniques that reflect their degree of familiarity with the use of limits, derivatives, and integrals, in simple and accessible contexts suitable for first-year students.

Not all of the participants have professional engineering experience, which may be the root of a phenomenon that has been less obvious in our previous papers. These participants list a number of external constraints that they use as rationales for *not* using applications or for not elaborating on them. We also observe that textbooks carry a good deal of weight for these participants. This phenomenon, already observed at the university level (e.g., Mesa & Griffiths, 2012), could be related to whether the teachers' background and experience is pertinent to the course they are teaching. The constraints identified by the participants include: 1) an overloaded syllabus, which they describe as creating pressure to avoid other material in order to cover the course content; this results in applications being excluded, with the further rationale that students will encounter them in other courses; 2) certain content is seen as a stepping stone to other content (such as limits leading to the study of derivatives), which may explain why some applications are set aside to leave room for the "important" content; 3) first-year classrooms are heterogeneous, with students on track to study different specialisations, presenting a challenge for teachers who feel they cannot cope with such a wide range of needs; 4) first-year students are limited in their knowledge, which makes it harder to use real-world applications.

Our study shows that while calculus teachers in engineering programmes acknowledge the influence of their background and professional experience on their teaching practices, they also believe that these practices, and the decisions behind them, are limited by certain constraints. We intend to pursue this line of research to better understand how these elements combine, in order to provide recommendations to postsecondary institutions for improving the training of mathematics teachers in these programmes.

References

Castela, C., & Romo-Vázquez, A. (2022). Towards an institutional epistemology. In R. Biehler, M. Liebendörfer, G. Gueudet, C. Rasmussen, & C. Winsløw (Eds.), *Practice-Oriented Research in Tertiary Mathematics Education* (pp. 621–648). Springer International Publishing. https://doi.org/10.1007/978-3-031-14175-1_30

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Faulkner, B., Johnson-Glauch, N., San Choi, D., & Herman, G. L. (2020). When am I ever going to use this? An investigation of the calculus content of core engineering courses. *Journal of Engineering Education, 109*(3), 402–423. https://doi.org/10.1002/jee.20344

González-Martín, A. S. (2021). $V_B - V_A = \int_A^B f(x)dx$. The use of integrals in engineering programmes: A praxeological analysis of textbooks and teaching practices in Strength of Materials and Electricity and Magnetism courses. *International Journal of Research in Undergraduate Mathematics Education, 7*(2), 211–234. https://doi.org/10.1007/s40753-021-00135-y

González-Martín, A. S., & Hernandes-Gomes, G. (2020). Mathematics in engineering programs: what teachers with different academic and professional backgrounds bring to the table. An institutional analysis. *Research in Mathematics Education, 22*(1), 67–86. https://doi.org/10.1080/14794802.2019.1663255

González-Martín, A. S., & Hernandes-Gomes, G. (2021). Applications in calculus for engineering. The cases of five teachers with different backgrounds. In M. Inprasitha, N., Changsri, & N., Boonsena (Eds.). *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 406–415). PME.

González-Martín, A. S., & Hernandes-Gomes. (submitted). Applications in Calculus courses in engineering: practices of teachers with different backgrounds. *Fifth Conference of the International Network of Didactic Research in University Mathematics* (INDRUM2024).

Kent, P., & Noss, R. (2003). *Mathematics in the university education of engineers*. The Ove Arup Foundation. https://www.ovearupfoundation.org/public/data/chalk/file/a/5/Kent-Noss-report.pdf

López-Díaz, M. T., & Peña, M. (2022). Encouraging students' motivation and involvement in STEM degrees by the execution of real applications in mathematical subjects: the population migration problem. *Mathematics, 10*(8), 1228. https://www.mdpi.com/2227-7390/10/8/1228

Mesa, V. & Griffiths, B. (2012). Textbook mediation of teaching: an example from tertiary mathematics instructors. *Educational Studies in Mathematics, 79*(1), 85-107. https://doi.or/10.1007/s10649-011-9339-9

Peters, J. (2022). Modifying exercises in mathematics service courses for engineers based on subject-specific analyses of engineering mathematical practices. In R. Biehler, M. Liebendörfer, G. Gueudet, C. Rasmussen, & C. Winsløw (Eds.), *Practice-Oriented Research in Tertiary Mathematics Education* (pp. 581–601). Springer International Publishing. https://doi.org/10.1007/978-3-031-14175-1_28

Quéré, P.-V. (2017). French engineers' training and their mathematical needs in the workplace: interlinking tools and reasoning. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Conference of the European Society for Research in Mathematics Education* (pp. 2233–2240). DCU Institute of Education and ERME

Slaught, H. E. (1908). Joint meetings of mathematicians and engineers at the University of Chicago. *Bulletin of the American Mathematical Society, 14*(6), 269–282.

BEYOND IMMEDIATE ERROR REPAIR: HOW TO SUPPORT TEACHERS' DECISION MAKING FOR ENHANCING UNDERSTANDING: AN EXPERIMENTAL STUDY

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How can diagnostic reports from formative assessment tools support teachers to derive decisions for enhancing students' understanding? In an experiment with 178 teachers, we compared two support conditions: The Error-Analysis report analyses student errors in detail, the Next-Goal report additionally explicates the next learning goal for this student. A quantitative analysis of teachers' task selections revealed that teachers using Next-Goal reports tended to select tasks focusing on more foundational learning goals than teachers using Error-Analysis reports and that they justified their selections significantly more often by referring to the essential learning goals. We conclude that Next-Goal reports can indeed better support teachers' targeted decision making.

INTRODUCTION: SUPPORT FOR DECISIONS AFTER ASSESSMENTS

After noticing (identifying and interpreting) student errors, many mathematics teachers need support for their decision making, i.e. to decide how to engage with these errors in ways that really can enhance students' understanding (Brodie, 2014). Formative assessment tools have the potential to support teachers in noticing their students' understanding and to derive adaptive and targeted decisions about enhancing it (Black & Wiliam, 2009), in particular when curriculum materials for enhancing understanding are provided that follow thoroughly designed learning trajectories (Siemon, 2019). But while different feedback modes *for students* in formative assessments have extensively been studied, little is known about how to best support *teachers* in their decision making based on formative assessments through diagnostic reports (Olsher et al., 2023). As diagnostic reports for teachers can have different designs, we ask:

To what extent does the design of diagnostic reports in formative assessments have an impact on teachers' decision making for enhancing students' understanding?

This research question was pursued in an experimental study with 178 practicing teachers for one selected concept in Grade 5: understanding of multiplication. We selected this basic concept, which students *should* have learned in Grade 2, because many students even in Grade 5 still hold only shallow knowledge of its meaning. Many German Grade 5-10 mathematics teachers, however, have not been sufficiently prepared for deepening this shallow knowledge (Prediger & Wischgoll, 2023), so that formative assessment bears some promise to support more targeted decisions.

2024. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 264–271). PME.

THEORETICAL BACKGROUND

Learning trajectory for multiplication

The long-term learning trajectory for multiplicative thinking spans several years (Siemon, 2019), in which one critical step is the learning goal of *understanding multiplication by the underlying unit structures* (Clarke & Kamii, 1996). This learning goal is focused on in the *Mastering Math* formative assessment and enhancement materials (Prediger, 2022). It is unpacked into three sub-learning goals and sequenced as follows: (1) counting in units (e.g., in pre-structured graphical representations of dice pictures as in the example Task 1 in Figure 1), (2) translating counting in units to symbolic multiplication, and (3) imposing unit structures onto the unstructured situations (starting with dot arrays), then connecting them to multiplication. This local learning trajectory for one or two sessions starts with eliciting students' experiences with prestructured figures like the dice, where talking about units ("five threes") is familiar to students (Task 1), who then only need to connect this to the symbolic multiplication. Based on this experience, teachers can focus mainly on imposing the unit structure in unstructured figures as in Task 5. This is needed in Task 7 to overcome a typical misconception for counting only edges in the "L-figure."

Teachers' noticing and decision making for multiplication

Teachers' noticing and decision making often focuses on procedural skills or superficial aspect, less on understanding (Prediger & Wischgoll, 2023; Siemon, 2019). As a background, studies have identified many teachers' short-term practices striving for immediate error repair instead of longer-term plans for enhancing the underlying understanding (Brodie, 2014; Prediger, 2022). For planning longer-term enhancement practices, expert teachers were found to unpack conceptual learning goals into finer-grained subgoals to sequence the subgoals into local learning trajectories and use them as diagnostic categories for noticing and decision making to successively enhance students' understanding along the learning trajectory (Gross et al., submitted).

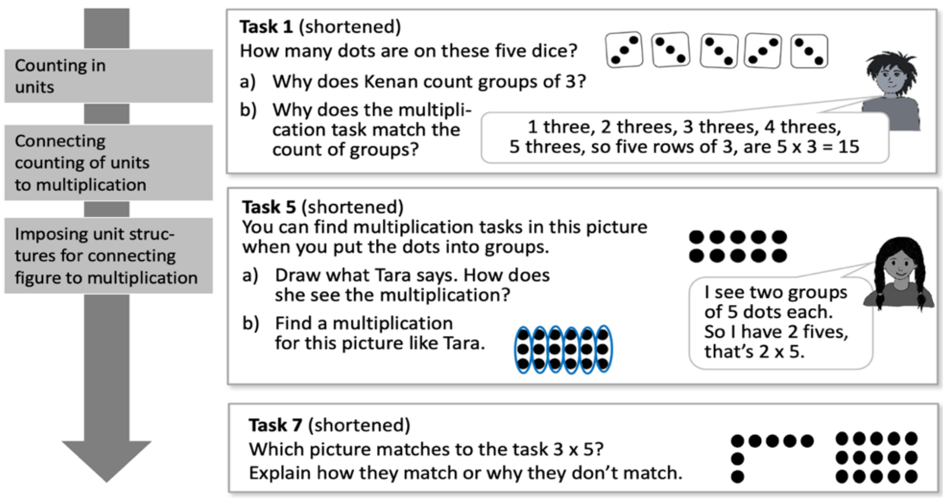

Figure 1: Local learning trajectory: Unpacking the learning goal for understanding multiplication into three subgoals with successive tasks for enhancing understanding

Diagnostic reports in formative assessment: Back to immediate error repair?

Formative assessment tools have been advertised for their potential to support teachers in noticing and adaptive decision making (Siemon, 2019; Black & Wiliam, 2009). So far, however, available diagnostic reports have tended to only include students' errors and their diagnoses (Olsher et al., 2023), here called "Error-Analysis reports." Since teachers were often reported to struggle with deriving appropriate goals for their students after error analysis (Brodie, 2014), we hypothesized that it might be more supportive for teachers when diagnostic reports contain the backgrounds of errors and also point to subsequent learning goals. In this paper, we test this hypothesis.

METHODS

Methods of data gathering for the quantitative support experiment

Dependent and independent variable. As the *dependent variable* of our support experiment, we assessed teachers' decision making for selecting enhancement tasks (Item 1a in Figure 2) and for justifying these selections (Item 1b). As the *independent variable*, two different support conditions were compared: *Error-analysis reports* analyze a student error in detail, and *Next-Goal reports* add an explicit articulation of the next learning goal for the student in view (diagnostic reports in Figure 2). As a *control variable*, we assessed how teachers analyzed a student error and reacted to it without diagnostic reports (Item 0, Prediger & Wischgoll, 2023). Items 0 and 1 treated the same error for two students (superficially translating a multiplication to a figure focusing only on edges standing for factors, without representing the unit structure). Teachers received support only in Item 1 by means of a detailed Error-Analysis report (and with a Next-Goal report, the additional information that unitizing was the next important learning goal for Lisa, who was not yet acquainted with multiplicative structures).

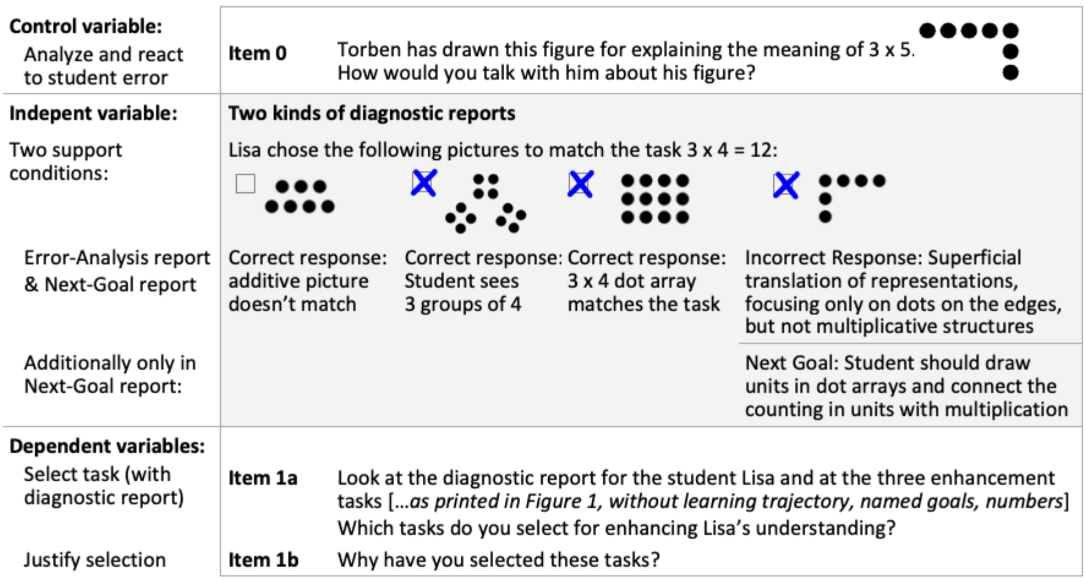

Figure 2: Research design: Variables and items in the support experiment

In Item 1a, teachers were asked to select one or more from 3 (unnumbered) tasks for enhancing Lisa's understanding (shortened in Figure 1). Teachers who favorize immediate error repair were expected to choose Task 7, whereas teachers searching for a safe foundation building on students' assets might use the dice in Task 1 (counting first in pre-structured units) before actively unitizing, i.e., imposing unit structures onto dot arrays for translating them into multiplications in Task 5 (in which dots first have to be gathered in groups before counting the new units). In Item 1b, we expect a justification of the selected tasks by referring to these learning goals involving unit structures.

Data gathering procedures. The diagnostic reports and items were integrated as a professional development (PD) activity starting with the third session of a synchronous online PD program on enhancing student understanding for basic arithmetic concepts. The first two Zoom PD sessions had treated general approaches for enhancing understanding and substantiated them for the place value system, but not yet for multiplication. Items 0-1 (from Figure 2) were administered in a Moodle course as the *Think* phase of a *Think-Pair-Share* routine.

Sample and cluster-randomized assignment. In total, 214 practicing teachers (of Grades 4-6 mathematics classes) participated in the PD session. Teachers from the same school were treated as clusters (of 1-4 teachers each), which were randomly assigned to the support conditions of Error-Analysis or Next-Goal reports. Our study sample consisted of 178 teachers who completed the activity and gave informed consent to use their Moodle data for research purposes. Their teaching experience ranged from 0-43 years, with median 9 years. Teachers in both conditions did not differ significantly in both teaching experience and the control variable: For the codes on Item 0, the t-test revealed $t(176) = 0.82$ and $p = .2066$. This means without the support of diagnostic reports, teachers in both conditions reacted similarly to student errors.

Methods of data analysis

Coding. In *Step 1*, teachers' task selections were coded (as they mostly chose either only 7, 5-7, 1-7, or 1/ 7, we coded the first selected task).

Code for addressed learning goal	Anchoring example
No conceptual learning goal addressed	"Lisa gets dot arrays and can search for the task."
Other conceptual learning goals addressed (not listed in Figure 1)	"Lisa can see that in the dot array, the factors can be switched."
Unit structures addressed as general learning goal	"The task addresses the principle of unit structure."
Unit structures addressed and unpacked into at least one learning subgoal	"She has to understand that each group must have the same amount of dots."

Table 1: Anchoring examples for coding teachers' justification

In *Step 2*, teachers' written decisions (reactions to student error in Item 0 and justifications for task selection in Item 1b) were deductively coded, based upon the coding scheme adapted from Prediger and Wischgoll (2023; see Table 1).

Statistical analysis. In *Step 3*, we tested our hypotheses (explicitly articulated below) by means of χ^2-tests and t-tests on significance levels of 5%.

RESULTS

Support potential for selecting tasks for enhancing understanding

With respect to teachers' decisions on selecting appropriate tasks, we hypothesized that

H1. Teachers supported by the Next-Goal report will select more targeted tasks than those with only the Error-Analysis report.

The upper part of Figure 3 shows which tasks the teachers selected in Item 1a for enhancing the understanding of the fictitious student Lisa. Under both conditions, 11% / 12%, resp., of the teachers went immediately to Task 7, which targets Lisa's misunderstanding directly (matching representations). These teachers strove for immediate error repair without realizing the better learning opportunities in Tasks 1 and 5 for the unit structures needed to represent the multiplication in the dot array and understand why the L-shaped figure does not match. Under both support conditions, the majority chose Task 5, focusing on the next learning goal (55 vs. 50 teachers, i.e., 56% vs. 62%), and 33% of those with Next-Goal reports (referred to as "NGR-supported") and 26% with Error-Analysis reports ("EAR-supported") selected Task 1, which targets the foundations of the unit structures by explicitly relating them to a student's asset. The overall numbers showed that both reports seemed to have supported the teacher in striving less for immediate error repairs than in other studies, but the difference between both groups was not significant in the χ^2-test (p = .61). So, H1 cannot be confirmed at all for the immediate error repair. For starting with foundations, we see a small but non-significant tendency.

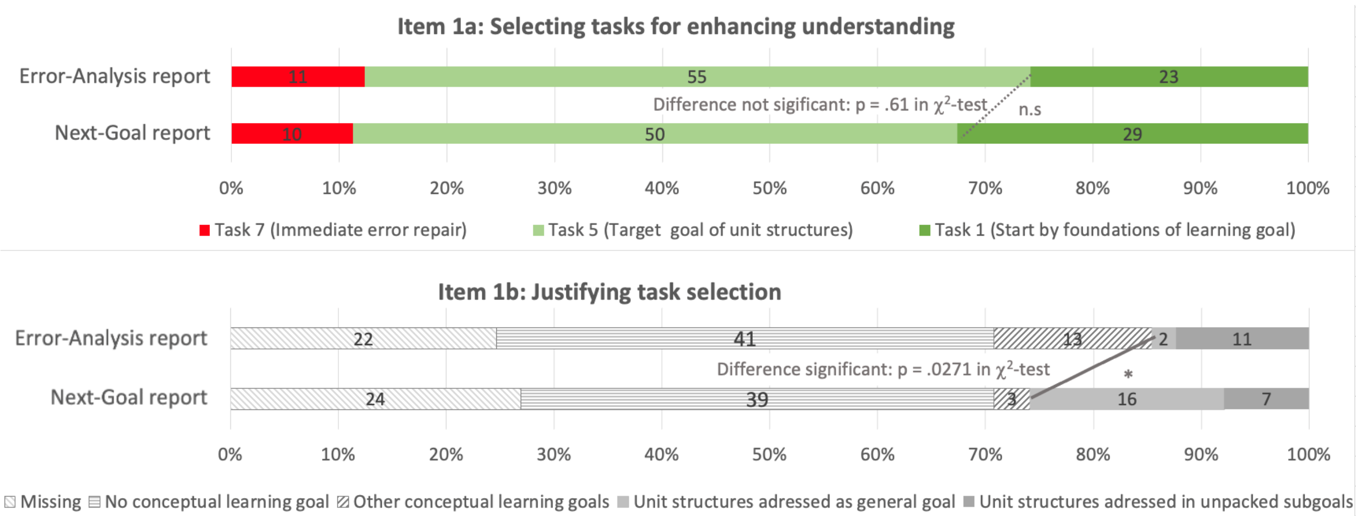

Figure 3: Differences in support conditions (Next-Goal and Error-Analysis reports) for selecting tasks and justifying task selections

Support potential for justifying task selections

With respect to teachers' goal-setting decisions expressed in their justifications of the task selections, we hypothesized that

H2. Teachers supported by a Next-Goal report will explicitly address the learning goal of multiplicative unit structures more often than those with an Error-Analysis report.

The lower part of Figure 3 reveals that none of the reports automatically led to addressing conceptual learning goals: 25 % / 27% did not write any justification (missing), and 44-46% justified their task selection without any conceptual learning goal. But large differences occurred among the other 30%: Whereas only 3 NGR-supported teachers referred to other conceptual learning goals that were not directly relevant for Lisa's learning progression (e.g., commutativity), 13 EAR-supported teachers did (3% vs. 15%). In contrast, 16 NGR-supported teachers addressed unit structures as a general goal that had not yet been unpacked into subgoals while only 2 EAR-supported teachers (18% vs. 2%) did. Interestingly, only 7 NGR-supported teachers unpacked the goal into at least one of the subgoals, but 11 in the less supported condition (8% vs. 12%) did. The χ^2-test for both codes addressing unit structures against the rest (striped in Figure 3) was significant (p = .0271).

We conclude that we have to refine the H2: While teachers supported by a Next-Goal report referred to the next learning goal in their justification significantly more often (unpacked or not unpacked), they less often adopted the unpacked subgoals offered in the report. Teachers with the Error-Analysis report, in turn, addressed the learning goal less often. Those few who did, however, more often unpacked it into refined subgoals. We assume this group comprised mainly those teachers not in need of support.

This finding led us to an analysis combining Items 1a and b: In the bar diagram of Figure 4, each bar represents a group of teachers who selected a particular task as starting task. Within each bar, we represent the relative frequency of addressing the unit structures goal, a significant distinction between conditions in Figure 3. Among those teachers who selected Task 1 or Task 7, no difference occurred between both support conditions: 0% of the teachers who went immediately to Task 7 explicitly addressed the learning goal of unit structures, while 21%/22% of those who selected Task 1 did.

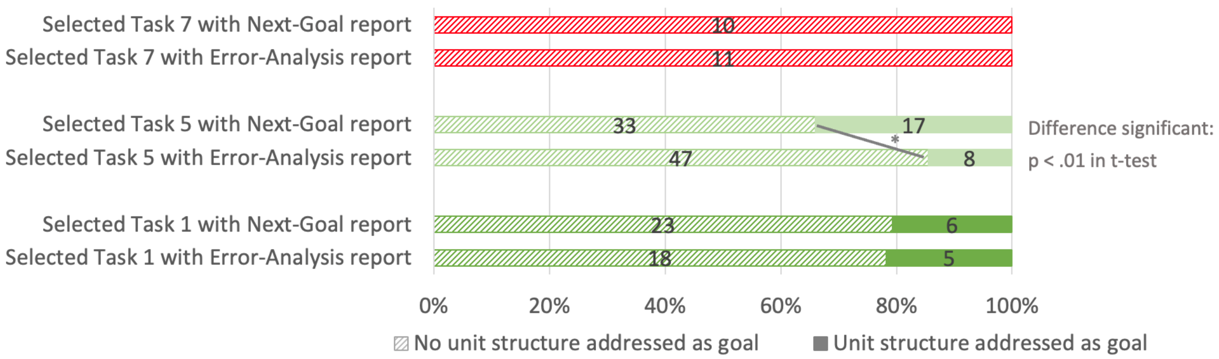

Figure 4: Relative distribution of addressed learning goals in justifications for each selected task under two support conditions (Next-Goal and Error-Analysis reports)

But within the group of teachers who selected the targeted Task 5, the relative distribution differed substantially: While 15% (8 of 55) of EAR-supported teachers named unit structures as goals, 34% (17 of 50) of NGR-supported teachers did. This difference is highly significant ($t(103) = -2.38$, $p < 0.01$) with a small effect size of $d = .46$.

SUMMARY AND DISCUSSION

In spite of a large body of research on mathematics teacher noticing and decision making with respect to student errors (Brodie, 2014), few studies have investigated how to *support* this decision making through appropriate and time-efficient formative assessment approaches (Siemon, 2019; Olsher et al., 2023). Our study analyzed whether teachers could be supported by providing advice for learning goals (Next-Goal reports) and not just the backgrounds of students' results (Error-Analysis reports).

The study faced methodological limitations in that only two items were analyzed, and some teachers problematized time constraints during data gathering, so in future research, more time must be planned. But still, the findings from 178 teachers provide valuable insights into support potentials of different diagnostic reports.

Analyzing the background of the error in the Error-Analysis report and suggesting three enhancement tasks (to select from) was already very substantial support: In Item 0 (without diagnostic report), only 17% of the teachers suggested any reaction explicitly alluding to making the fictitious student Torben aware of the underlying unit structures, and these frequencies resonate with a previous study (Prediger & Wischgoll, 2023). But with the reports, 88% / 89% of the teachers, resp., selected tasks that helped them work on learning goals that were suitable for Lisa's error.

Hypothesis H1 (assuming differences between two support conditions for teachers' task selection) could not be confirmed, as only a slight but non-significant tendency occurred to choose Task 1 more often than Task 5. For selecting the tasks, the Next-Goal did seem to hardly help more than the Error-Analysis report.

Hypothesis H2 (assuming differences between two support conditions for teachers' justifications of task selection), on the other hand, could be substantiated in a refined manner. As expected, effects of diagnostic reports were not explicitly visible for all teachers (as they might have good goals in mind without articulating them in written justifications). But we observed that significantly more teachers with Next-Goal reports explicitly alluded to learning goals of unit structures (26% vs. 13%) than those with Error-Analysis reports (Figure 3). The refined analysis in Figure 4 revealed unequal distributions for groups of teachers according to their selections of tasks. It was remarkable to us that the support effect of the Next-Goal report was entirely concentrated on teachers choosing Task 5 as their starting task: In this group, more than twice as many teachers explicitly referred to unit structures (34% vs. 15%, a highly significant difference). Given the state of research, we might explain this concentration as follows: More teachers who were completely guided by short-term intentions to repair errors (by immediately going to Task 7) seemed to be less attentive to the error analysis and suggested next learning goals than those who sought to enhance the underlying understanding, which resonates with observations by Brodie (2014). In contrast, teachers who believed in working through the complete learning trajectory as suggested by the materials, might not have needed to read the advice, but rather started with Task 1 anyway (as also found in an interview study, Gross et al., submitted). The support effect, therefore, was concentrated on a third group. In the future, we intend to investigate in how far the advice about next learning goals has an impact on teachers' moderation strategies when working on the task with the student in a well-aligned way.

Overall, the support experiment provided evidence that at least some teachers can indeed be supported without time- and money-intensive professional development programs (Prediger, 2022), as subtle differences in the design of a diagnostic report can already have significant effects. However, the design needs to be further developed to reach more of the teachers (Olsher et al., 2023).

Acknowledgement. We thank U. Kroehne (DIPF), C. Hankeln, L. Voss, and K. Rösike (TU Dortmund University) for productive cooperation on our Mastering Math Online-Check. It is conducted within the project SchuMaS, financially funded by the German Ministry of Education and Research (BMBF grant no. SMS2101L-01PR2101C to S. Prediger).

References

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Ed. Assessment, Evaluation & Acc.*, *21*(1), 5–31. https://doi.org/10.1007/s11092-008-9068-5

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Ed. Studies in Mathematics*, *85*(2), 221–239. https://doi.org/10.1007/s10649-013-9507-1

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education*, *27*(1), 41–51. http://www.jstor.com/stable/749196

Gross, S., Hankeln, C., Rösike, K.-A., & Prediger, S. (submitted). How do expert and novice teachers monitor and enhance student understanding? Submitted manuscript.

Olsher, S., Chazan, D., Drijvers, P., Sangwin, C., & Yerushalmy, M. (2023). Digital assessment and the "machine". In B. Pepin, G. Gueudet, & J. Choppin (Eds.), *Handbook of Digital Resources in Mathematics Education* (pp. 1-27). Springer.

Prediger, S. (2022). Implementation research as a task for subject-matter education disciplines [in the Mastering Math project]. *RISTAL Research in Subject-matter Teaching and Learning, 5*(1), 4–23. https://doi.org/10.23770/rt1852

Prediger, S., & Wischgoll, A. (2023). Unit structures rarely articulated: Teachers' explanations of meanings of multiplication. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of PME46* (Vol. 4, pp. 59–66). PME.

Siemon, D. (2019). Knowing and building on what students know: The case of multiplicative thinking. In D. Siemon, T. Barkatsas, & R. Seah (Eds.), *Researching and Using Progressions (Trajectories) in Mathematics Education* (pp. 6–31). Brill.

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